Surface quotient singularities and bigness of the cotangent bundle: Part I

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Abstract

We investigate the components determining bigness of the cotangent bundle Ω^1_X of smooth models X in the birational class $\mathcal{Y}$ of an orbifold surface of general type Y, with a focus on the contribution given by the singularities of Y. A criterion for bigness of Ω^1_X is given involving only topological and singularity data on Y. We single out a special case, the Canonical Model Singularities (CMS) criterion, when Y is the canonical model of $\mathcal{Y}$. We study the singularity invariants appearing in the criterion and determine them for A_n singularities. Knowledge of these invariants for A_n singularities allows one to evaluate the $(c_2, c_1^2)-$geographical range of the CMS criterion and compare it to other criteria. We obtain new examples of resolutions X of hypersurfaces $Y \subset \mathbb{P}^3$ (with lower degrees) and of cyclic covers Y of $\mathbb{P}^2$ branched along line arrangements with Ω^1_X big.

Keywords: A_n singularities, symmetric differentials, big cotangent bundle, extension results, surfaces of general type

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1 Introduction

A smooth projective surface X has big cotangent bundle Ω^1_X if

$$h^0_\Omega(X) := \lim_{m\to\infty} \frac{h^0(X, S^m\Omega^1_X)}{m^3} \neq 0,$$

i.e., the symmetric pluri-genera $P^S_m(X) := h^0(X, S^m\Omega^1_X)$ has the maximal growth order possible with respect to m for $\dim X = 2$. The symmetric plurigenera are birational invariants and hence the cotangent bundle being big is a birational property (among smooth representatives).

Bigness of Ω^1_X is a manifestation of positivity properties of Ω^1_X and it implies that the canonical line bundle $\wedge^2\Omega^1_X$ is big, i.e., X is a surface of general type, see for example [1]. A motivation to study bigness of the cotangent bundle comes from the connection between positivity properties of the cotangent bundle and the hyperbolicity properties of projective varieties. A de facto connection is that Ω^1_X ample implies X is Kobayashi hyperbolic [2]. A conjectural connection is the Green-Griffiths-Lang (GGL) conjecture stating that: A projective variety X of general type has a proper subvariety $Z \subset X$ such that all entire curves of X are contained in Z. More pertinent to this work – a consequence of the works of Bogomolov [3] and McQuillan [4] – is that a surface of general type with big Ω^1_X satisfies the GGL conjecture. For surveys on this topic see for example [5] and [6].

Let X be a smooth surface of general type. Riemann-Roch and Bogomolov's vanishing, $h^2(X, S^m\Omega^1_X) = 0$, $m > 2$ (a consequence of the semi-stability properties of the tangent bundle T_X for minimal surfaces of general type [7] and [8], see also Lemma 1), give that for any smooth surface of general type X:

$$h^0_\Omega(X) = h^1_\Omega(X) + \frac{1}{3!}s_2(X),$$

where $s_2(X) = c_1^2(X) - c_2(X)$ is the 2nd Segre number and $h^1_\Omega(X) := \lim_{m\to\infty} h^1(X, S^m\Omega^1_X)/m^3$ (this limit exists, see section 3.1). In [7], Bogomolov concluded that $s_2(X) > 0$ implies Ω^1_X is big and showed how the bigness of Ω^1_X implies the boundness of the family of curves of fixed geometric genus in X. While $s_2(X)$ is easy to get a hold on, the invariant $h^1_\Omega(X)$ is difficult to determine and depends on the complex structure, as it can jump under deformation, this was described in [9].

When X is a resolution of an orbifold surface Y (normal surface with only quotient singularities) of general type, we derive a lower bound on $h^1_\Omega(X)$ using singularity data of Y, and obtain a criterion for bigness of Ω^1_X. Each singularity gives an independent contribution to $h^1_\Omega(X)$. Given a normal surface singularity y, set:

$$h^1_\Omega(y) := \liminf_{m\to\infty} \frac{h^0(U_y, R^1\sigma_* S^m\Omega^1_{\tilde{U}_y})}{m^3},$$

where U_y is a neighborhood germ of the surface singularity y and $\sigma : \tilde{U}_y \to U_y$ its minimal resolution. We call $h^1_\Omega(y)$ the 1-st cohomological Ω-asymptotics of the surface singularity y.

Theorem 1 *(QS-Bigness Criterion) Let X be a surface of general type which is the minimal resolution of a surface Y with only quotient singularities. Then Ω^1_X is big if:*

$$\sum_{y\in \mathrm{Sing}(Y)} h^1_\Omega(y) + \frac{s_2(X)}{3!} > 0,$$

The approach taken in this paper relies on the theory developed by Wahl [10], Blache [11] and Langer [12] for the Chern classes (local and global) and the asymptotic Riemann-Roch formulas for orbifold vector bundles. We also require the use of the results on the semi-stability of the tangent sheaf of singular varieties due to Guenancia and Kobayashi (see [13] theorem A and [14]) to derive the vanishing $H^2(Y, \hat{S}^m\Omega^1_Y) = 0$, Lemma 1, where $\hat{S}^m\Omega^1_Y = [\sigma_*(S^m\Omega^1_X])^{\vee\vee}$ is the orbifold m-symmetric power of Ω^1_Y, with X and Y as in the theorem.

An equivalence class $\mathcal{X}$ of normal projective surfaces under birational equivalence relation will be referred to as a surface birational class. The cotangent bundle $\Omega^1_{\mathcal{X}}$ is big if any (and hence all) of the smooth models X of $\mathcal{X}$ has Ω^1_X big. A surface birational class of general type $\mathcal{X}$ has two special models, X_{can} and X_{min} respectively the canonical and the minimal models.

The Chern numbers of $\mathcal{X}$ will the Chern numbers of its minimal model, as an illustration, $s_2(\mathcal{X}) := s_2(X_{\mathrm{min}})$. We call *the localized component* (at the singularities) of $h^1_\Omega(\mathcal{X}) := h^1_\Omega(X_{\mathrm{min}})$:

$$Lh^1_\Omega(\mathcal{X}) = \sum_{x\in \mathrm{Sing}(X_{\mathrm{can}})} h^1_\Omega(x)$$

There is a natural application of the QS-Bigness Criterion towards bigness of $\Omega^1_{\mathcal{X}}$.

Corollary 1 *(CMS-bigness criterion) Let $\mathcal{X}$ be a birational class of surfaces of general type. Then the cotangent bundle $\Omega^1_{\mathcal{X}}$ is big if*

$$Lh^1_\Omega(\mathcal{X}) + \frac{s_2(\mathcal{X})}{3!} > 0$$

To ascertain the scope of the CMS-criterion we need a hold on $Lh^1_\Omega(\mathcal{X})$. The starting point is to investigate 1-st cohomological Ω-asymptotics, $h^1_\Omega(y)$, for canonical singularities. These invariants were previously unknown with the exception of $h^1_\Omega(A_1)$, when y is an A_1 singularity [15]. We note also that in one of the authors thesis [16], using an earlier version of the method appearing in this paper, the cases A_n, $n = 2, 3, 4, 8, 12$ and 16 were treated. In this work, we give a lower bound for $h^1_\Omega(y)$ involving relative local Chern numbers when y is a canonical singularity, and in Part II we determine a closed formula for $h^1_\Omega(A_n)$ for all A_n singularities.

Our approach to find $h^1_\Omega(y)$ uses the singularity invariants $\chi(y,m)$, $\chi_{\text{orb}}(y,m)$ and $\mu(y,m)$ appearing as correction terms in the relations between the Euler characteristics (with X and Y as in Theorem 1):

$$\chi(Y, \hat{S}^m\Omega^1_Y) = \chi(X, S^m\Omega^1_X) + \sum_{y\in \text{Sing(Y)}} \chi(x,m)$$
$$\chi(Y, \hat{S}^m\Omega^1_Y) = \chi_{\text{orb}}(Y, \hat{S}^m\Omega^1_Y) + \sum_{y\in \text{Sing(Y)}} \mu(y,m)$$
$$\chi(X, S^m\Omega^1_X) = \chi_{\text{orb}}(Y, \hat{S}^m\Omega^1_Y) + \sum_{y\in \text{Sing(Y)}} \chi_{\text{orb}}(y,m)$$

and the resulting relation:

$$\chi(y,m) := \hbar^0(y,m) + h^1(y,m) = \mu(y,m) - \chi_{\text{orb}}(y,m), \tag{+}$$

where $h^1(y,m) := h^0(U_y, R^1\sigma_* S^m\Omega^1_{U_y})$ and $\hbar^0(y,m) = \dim[H^0(\tilde{U_y}\setminus E, S^m\Omega^1_{\tilde{U_y}})/H^0(\tilde{U_y}, S^m\Omega^1_{\tilde{U_y}})]$, where $(\tilde{U_y}, E)$ is the minimal resolution of the neighborhood germ (U_y, y).

The following key relation follows from the asymptotics of $\chi_{\text{orb}}(y,m)$ and $\mu(y,m)$ [10]:

$$\lim_{m\to\infty} \frac{\hbar^0(y,m) + h^1(y,m)}{m^3} = \frac{c_2(y) - c_1^2(y)}{3!}$$

where $c_2(y)$, $c_1^2(y) \in \mathbb{Q}$ are the relative local Chern numbers of y. This relation tell us that $h^1_\Omega(y)$ can be derived from the asymptotics of $\hbar^0(y,m)$.

In the case of canonical singularities, local cohomology and the work of Karras [17] (see also [18]) on the cohomology with compact supports on neighborhoods of exceptional sets allows us to derive the following lower bounds:

Theorem 2 *Let (Y,y) be the neighborhood germ of a canonical surface singularity.*

1. *For all $m \geq 0$, $\hbar^0(y,m) \leq h^1(y,m)$*
2. *$h^1_\Omega(y) \geq \frac{c_2(y)}{2\cdot 3!}$*

The comparison $h^1_\Omega(y) \geq \frac{c_2(y)}{2\cdot 3!}$ is pertinent because it shows that CMS-criterion is stronger that the bigness criterion appearing in [19]. For A_n singularities though, we know both sides and have that $\psi(n) := \frac{h^1_\Omega(A_n)}{\frac{c_2(A_n)}{2\cdot 3!}}$ is increasing, $\psi(1) = \frac{32}{27}$ and $\lim_{m\to\infty}\psi(n) = 2$, see Corollary 3.

In the case of A_n singularities we can determine $h^1_\Omega(A_n)$. In Theorem 1(II) (Theorem 1 of Part II), we determine $\hbar^0(A_n, m)$. We show that for each n and m, $\hbar^0(A_n,m)$ is given by a weighted lattice sum over a polygon $\mathcal{P}_n(m)$,

$$\hbar^0(A_n,m) = \sum_{\substack{\mathbf{x}=(x_1,x_2)\in\mathcal{P}_n(m)\cap\mathbb{Z}^2 \\ x_1+(n+1)x_2\equiv m \bmod 2}} h_{n,m}(\mathbf{x})$$

and if we fix n, then $\hbar^0(A_n, m)$ is a quasi-polynomial in m (the coefficients in powers of m are periodic functions of m) of degree 3 with constant leading coefficient $\hbar^0_\Omega(A_n)$ for which we present a closed formula in n. We then obtain in Theorem 2(II) the formula:

$$h^1_\Omega(A_n) = \frac{n^5 + 19n^4 + 83n^3 + 137n^2 + 80n}{6(n+1)^2(n+2)^2} - \frac{4}{3}\sum_{k=1}^{n}\frac{1}{k^2},$$

To investigate the strength of the CMS criterion it is instructive to compare it to the criterion by Roulleau and Rousseau in [19], from now on denoted by the RR-criterion. The RR-criterion for bigness of $\Omega^1_{\mathcal{X}}$ uses the "stacky" framework of orbifod structures attached to X_{can} inspired by the work of Campana [20], but it can be reformulated using our framework as:

$$\sum_{x \in \text{Sing}(X_{\text{can}})} \frac{c_2(x)}{2 \cdot 3!} + \frac{s_2(\mathcal{X})}{3!} > 0 \implies \Omega^1_{\mathcal{X}} \text{ is big}$$

The distinction between the two criteria lies in how each assesses the impact of each singularity in $h^1_\Omega(\mathcal{X})$. First, we remark that Theorem 2(b) implies that the CMS-criterion always holds if the RR-criterion holds. But to comprehend the full strength of the CMS-criterion the formula for $h^1_\Omega(A_n)$ is required. This formula shows that the ratio between the two criteria singularity contributions, $h^1_\Omega(A_n)$ over $\frac{c_2(A_n)}{2\cdot 3!}$, grows with n, with $32/27$ for $n = 1$ and approaching 2 as $n \to \infty$. The striking impact of this difference is seen in the following paragraphs.

Let GT be the set of all birational classes of surfaces of general type and CGT be the set of Chern number pairs $CGT := \{(c_2(\mathcal{X}), c_1^2(\mathcal{X})) | \mathcal{X} \in GT\}$, where $c_i(\mathcal{X}) := c_i(X_{\min})$. Given a criterion C define the excluded range of C by:

$$ER(C) = \{(a, b) \in CGT | \nexists \mathcal{X} \in GT \text{ with } (c_2(\mathcal{X}), c_1^2(\mathcal{X})) = (a, b) \text{ satisfying } C\}$$

The reason for a pair of Chern numbers (a, b) to belong to $ER(\text{CMS})$ or to $ER(\text{RR})$ must come from bounds on the possible singularities of the canonical model X_{can} of $\mathcal{X}$ with $(c_2(\mathcal{X}), c_1^2(\mathcal{X})) = (a, b)$. The are two such general bounds:

$$\text{M-bound (Miyaoka [21]):} \quad \sum_{x \in \text{Sing}(X_{\text{can}})} c_2(x) \le c_2(\mathcal{X}) - \frac{1}{3}c_1^2(\mathcal{X}) \qquad (*)$$

$$\text{H-bound (Standard Hodge theory):} \quad \rho_{(-2)}(\mathcal{X}) \le \frac{1}{6}(5c_2(\mathcal{X}) - c_1^2(\mathcal{X})) + b_1(\mathcal{X}) - 1, \qquad (**)$$

where $\rho_{(-2)}(\mathcal{X}) :=$ # of (-2)-curves on $X_{\min}$ and $b_1(\mathcal{X}) := b_1(X_{\min})$.

The M-bound explicitly bounds the singularity contribution on the RR-criterion, making its impact on $ER(\text{RR})$ straightforward: ([19]) $(a, b) \in CGT$ must belong to $ER(\text{RR})$ if $\frac{b}{a} \le \frac{3}{5}$.

On the other hand, the restriction imposed by the M-bound on the CMS-criterion concerning the excluded range is minor. The only pairs in CGT that the M-bound forces to be in $ER(CMS)$ is the finite collction with $c_1^2 = 1, 2$. This contrasts with the infinite collection pairs forced to belong to $ER(\text{RR})$ by the M-bound, as mentioned in the paragraph above.

The H-bound is more restrictive than the M-bound for the CMS-criterion when considering birational classes $\mathcal{X}$ with $b_1(\mathcal{X}) = 0$. The singularity contribution in the CMS-criterion is the highest if all the (-2)-curves allowed on $X_{\min}$ come from a single singularity. The bound on $\rho_{(-2)}(\mathcal{X})$ given by the H-bound is stronger than the one implied by the M-bound when $c_1^2 < c_2$, if $b_1(\mathcal{X}) = 0$, see Section 2.7.

Set:

$$\alpha_{_{\text{Big}}} = \inf \left\{ \frac{c_1^2(\mathcal{X})}{c_2(\mathcal{X})} | \ \mathcal{X} \in GT \text{ with } \Omega^1_{\mathcal{X}} \text{ big} \right\}$$

The impact, described above, of M-bound on the $ER(RR)$ shows that the RR-criterion can not show $\alpha_{_{\text{Big}}} < 3/5$.

Taking into account both bounds and considering canonical models with only singularities of type A, we obtain: the H-bound and M-bound together can only possibly force $(a,b) \in ER(\text{CMS})$ if $b \leq \frac{1}{5}a$. Hence the CMS-criterion implies:

Theorem 3 *(CMS-criterion range bounds) If a birational class $\mathcal{X}$ of surfaces of general type has its canonical model with only singularities of type A, then $\mathcal{X}$ can not satisfy the CMS-criterion if $c_1^2 \leq \frac{c_2}{5}$ holds.*

Moreover, the CMS-criterion implies that the Miyaoka (*) and the Hodge theoretical (**) bounds cannot rule out α_{Big} to be as low as 1/5.

It is well known that smooth hypersurfaces $H_d \subset \mathbb{P}^3$ of degree $d \geq 1$ have no symmetric differentials [22], on the other hand in [9] it was shown that if $d \gg 0$ there are surfaces X deformation equivalent to H_d with Ω^1_X big. An application of the CMS-criterion gives the following result concerning:

Problem: what is $d_{\min} = \min\{d | \exists X \text{ deformation of } H_d \text{ with } \Omega^1_X \text{ big}\}$?

Theorem 4 *For $d \geq 8$ there are surfaces X deformation equivalent to smooth hypersurfaces $H_d \subset \mathbb{P}^3$ of degree d with big cotangent bundle.*

It follows that $d_{\min} \leq 8$ and $\alpha_{_{\text{Big}}} \leq \frac{c_1^2}{c_2}(H_8) = \frac{8}{19}$ $(< \frac{3}{5})$. The surfaces X appearing in the theorem can be obtained as the minimal resolution of hypersurfaces $Y \subset \mathbb{P}^3$ of degree d with only A_{d-1} singularities, where Y are the cyclic covers of degree d of $\mathbb{P}^2$ branched along d lines in general position. No known obstruction exists to prevent the CMS-criterion from achieving $d_{\min} = 6$, while it follows from Theorem 3 that $d_{\min} = 5$ can not be achieved with resolutions of hypersurfaces $Y \subset \mathbb{P}^3$ with only singularities of type A.

Another application concerns the minimal resolutions of cyclic covers of $\mathbb{P}^2$ branched along line arrangements in general position:

Theorem 5 *The CMS-criterion guarantees bigness of the cotangent bundle and hence the GGL-conjecture to hold for all the minimal resolutions $Y_{n,d}$ of general type that are cyclic covers of $\mathbb{P}^2$ of degree n and branched on $d = \nu n$ lines in general position, with the exception of list in the table below.*

Table 1: The only pairs (n, ν) for which $\Omega^1_{Y_{n,\nu n}}$ is not big

ν	1	2, 3	$4 \leq \nu \leq 7$	$\nu \geq 8$
n	5, 6, 7	3, 4	2, 3	2

We also address the question about the existence of symmetric differentials of a given degree m on deformations of smooth hypersurfaces $H_d \subset \mathbb{P}^3$ of degree $d \geq 5$. To this end, we need the formula for $\hbar^0(A_n, m)$ obtained in Theorem 1(II) and the formula for local Euler characteristic $\chi(A_n, m)$. The latter follows easily from the singularity invariant $\mu(A_n, m)$ using (+).

As mentioned earlier, we have $\chi(Y,\hat{S}^m\Omega^1_Y)=\chi_{\mathrm{orb}}(Y,\hat{S}^m\Omega^1_Y)+\sum_{y\in\mathrm{Sing(Y)}}\mu(y,m)$. We find the $\mu(y,m)$ invariants in the case of A_n singularities to be:

Theorem 6 *The $\mu(A_n,m)$ are given by the formulas, where $r\equiv m\,(\mathrm{mod}\,n+1)$ and $0\le r\le n$:*

$$\mu(A_n,m)=\begin{cases}\frac{(-1)^m}{4}\lfloor\frac{m+1}{n+1}\rfloor+\alpha_n(r) & n\ \textit{odd}\\ \alpha_n(r) & n\ \textit{even},\end{cases}$$

with $\alpha_n(r)$ given by

$$\alpha_n(r)=\begin{cases}\frac{r+1}{12(n+1)}\big(2r(r+2)+n(n+2)\big)-\frac{2r^2+4r+1-(-1)^r}{8} & r\neq n\\ 0 & r=n.\end{cases}$$

In particular, for fixed n, $\mu(A_n,m)$ as function of m is bounded if n is even and $|\mu(A_n,m)|$ is bounded by a linear function if n is odd.

2 Orbifold symmetric powers of Ω^1_X, surface quotient singularities invariants and their asymptotics

2.1 Quotient singularities and orbifold vector bundles on surfaces

In this work, a normal projective surface X is called an orbifold surface if the singularities of X are quotient (log terminal) singularities. In the literature an orbifold surface is also called a $\mathbb{Q}$-surface (e.g. [23]) or a normal V-surface (e.g. [11]).

The germ of an isolated quotient surface singularity (X,x) is biholomorphic to a quotient $(\mathbb{C}^2,0)/G_x$, with subgroup $G_x\subset GL_2(\mathbb{C})$ finite and small; G_x is the local fundamental group. Canonical surface singularities are the quotient singularities with $G_x\subset SL_2(\mathbb{C})$, and their classification is the same as that of simply connected simple Lie groups (hence the ADE nomenclature).

2.1.1 Resolution and smoothing pairs

Associated to a germ of an isolated quotient surface singularity (X,x), we have the resolution/smoothing pair:

$$\begin{array}{ccc} & & (\mathbb{C}^2,0)\\ & & \downarrow{\scriptstyle\pi}\\ (\tilde{X},E) & \xrightarrow{\sigma} & (X,x)\end{array}\tag{1}$$

with $\pi:(\mathbb{C}^2,0)\to(X,x)$, the quotient map by the local fundamental group, called the local smoothing of (X,x) and $\sigma:(\tilde{X},E)\to(X,x)$ a good resolution of (X,x) where $(\tilde{X},E)$ is the germ of a neighborhood of the exceptional locus E with E consisting of smooth curves intersecting transversally.

2.1.2 Orbifold vector bundles and reflexive sheaves

The sub-class of coherent sheaves on orbifold surfaces relevant to our goals consists of orbifold vector bundles (also called $\mathbb{Q}$-vector bundles or locally V-free sheaves). In the surface case, this sub-class coincides with the class of reflexive sheaves (cf. Section 2 of either [23] or[11]).

A reflexive coherent sheaf $\mathcal{F}$ on an orbifold X is called an *orbifold vector bundle* if each point $x\in X$ has a neighborhood germ (X,x) and a smoothing $\pi:(\mathbb{C}^n,0)\to(X,x)$ with a locally free sheaf $\mathcal{F}'$ such that

$$\mathcal{F}|_{(X,x)}=(\pi_*^{G_x})\mathcal{F}',$$

where $(\pi_*^{G_x})\mathcal{F}'$ is the maximal subsheaf of $\pi_*\mathcal{F}'$ on which G_x acts trivially. In dimension 2, this condition always holds by setting $\mathcal{F}' = (\pi^*\mathcal{F}|_{(X,x)})^{\vee\vee}$.

Associated to a reflexive sheaf $\mathcal{F}$ on the quotient surface germ (X, x) there are also locally free sheaves $\tilde{\mathcal{F}}$ on $(\tilde{X}, E)$, which are not uniquely determined, satisfying $\mathcal{F} \cong (\sigma_*\tilde{\mathcal{F}})^{\vee\vee}$.

2.1.3 Orbifold bundles associated to the cotangent bundle

Let X be an orbifold surface, $\sigma : \tilde{X} \to X$ a good resolution of X and $i : X_{\text{reg}} \hookrightarrow X$ the natural inclusion of the regular part of X. We are interested in the following orbifold vector bundles:

i) the cotangent and tangent bundles of X, which are respectively the reflexive sheaves $\Omega^1_X := i_*(\Omega^1_{X_{\text{reg}}})$ and its dual $T_X := i_*(T_{X_{\text{reg}}})$.
ii) the canonical bundle $\mathcal{O}(K_X)$ $(\cong (\bigwedge^2 \Omega^1_X)^{\vee\vee})$.
iii) the orbifold m-symmetric powers of the cotangent bundle

$$\hat{S}^m\Omega^1_X := (S^m\Omega^1_X)^{\vee\vee} \cong i_*(S^m\Omega^1_{X_{\text{reg}}}) \cong (\sigma_* S^m\Omega^1_{\tilde{X}})^{\vee\vee}.$$

Note that $\hat{S}^m\Omega^1_X$ is not necessarily isomorphic to $S^m\Omega^1_X$.

2.2 Local holomorphic Euler characteristic

Let X be a compact normal orbifold complex surface, $\tilde{X} \xrightarrow{\sigma} X$ be a good resolution, $\tilde{\mathcal{F}}$ and $\mathcal{F}$ sheaves such that $\tilde{\mathcal{F}}$ is locally free on $\tilde{X}$, and $\mathcal{F} = (\sigma_*\tilde{\mathcal{F}})^{\vee\vee}$ a reflexive sheaf on X. The relation between the holomorphic Euler characteristic of $\tilde{\mathcal{F}}$ and that of the reflexive sheaf $\mathcal{F} = (\sigma_*\tilde{\mathcal{F}})^{\vee\vee}$ is given by

Proposition 7 ([10] or [11, 3.9]) *With the notation of the previous paragraph for X, $\tilde{X}$, $\mathcal{F} = (\sigma_*\tilde{\mathcal{F}})^{\vee\vee}$ and $\tilde{\mathcal{F}}$, then:*

$$\chi(X, \mathcal{F}) = \chi(\tilde{X}, \tilde{\mathcal{F}}) + \sum_{x \in Sing(X)} \chi(x, \tilde{\mathcal{F}}) \tag{2}$$

$$\chi(x, \tilde{\mathcal{F}}) := h^0(U_x, (\sigma_*\tilde{\mathcal{F}})^{\vee\vee}/\sigma_*\tilde{\mathcal{F}}) + h^0(U_x, R^1\sigma_*\tilde{\mathcal{F}})$$

or equivalently:

$$\chi(x, \tilde{\mathcal{F}}) = \dim[H^0(\tilde{U}_x \setminus E_x, \tilde{\mathcal{F}})/H^0(\tilde{U}_x, \tilde{\mathcal{F}})] + h^1(\tilde{U}_x, \tilde{\mathcal{F}}) \tag{3}$$

where the U_x are Stein neighborhoods (can be chosen to be affine if X is projective) such that $U_x \cap \operatorname{Sing}(X) = \{x\}$. The $\chi(x, \tilde{\mathcal{F}})$ is called the local holomorphic Euler characteristic of $\tilde{\mathcal{F}}$ at x.

Proof For the convenience of the reader, we provide a sketch of the argument.The Leray-Serre spectral sequence for the sheaf $\tilde{\mathcal{F}}$ and the morphism $\tilde{X} \xrightarrow{\sigma} X$ gives:

$$0 \to H^1(X, \sigma_*\tilde{\mathcal{F}}) \to H^1(\tilde{X}, \tilde{\mathcal{F}}) \to H^0(X, R^1\sigma_*\tilde{\mathcal{F}}) \to H^2(X, \sigma_*\tilde{\mathcal{F}}) \to H^2(\tilde{X}, \tilde{\mathcal{F}}) \to 0$$

Moreover, $H^0(X, \sigma_*\tilde{\mathcal{F}}) \simeq H^0(\tilde{X}, \tilde{\mathcal{F}})$. The previous in conjunction with $R^1\sigma_*\tilde{\mathcal{F}}$ being supported at the singularities of X gives that:

$$\chi(X, \sigma_*\tilde{\mathcal{F}}) = \chi(\tilde{X}, \tilde{\mathcal{F}}) + \sum_{x \in \operatorname{Sing}(X)} h^0(U_x, R^1\sigma_*\tilde{\mathcal{F}})$$

To conclude, the cohomology long exact sequence for:

$$0 \to \sigma_*\tilde{\mathcal{F}} \to \mathcal{F} \to \mathcal{F}/\sigma_*\tilde{\mathcal{F}} \to 0$$

and $\mathcal{F}/\sigma_*\tilde{\mathcal{F}}$ again being supported at the singularities of X gives :

$$\chi(X, \sigma_*\tilde{\mathcal{F}}) = \chi(X, \mathcal{F}) - \sum_{x \in \operatorname{Sing}(X)} h^0(U_x, \mathcal{F}/\sigma_*\tilde{\mathcal{F}})$$

Also observe that the reflexive nature of $\mathcal{F} = (\sigma_*\tilde{\mathcal{F}})^{\vee\vee}$ gives that $H^0(U_x, \mathcal{F} = H^0(U_x \setminus \{x\}, \mathcal{F}) \simeq H^0(\tilde{U}_x \setminus E_x, \tilde{\mathcal{F}})$ □

2.3 Surface singularity invariants $\hbar^0(x,m)$ and $h^1(x,m)$

The study of symmetric differentials on resolutions of orbifold surfaces requires a handle on the following surface invariants.

Definition 1 Let $\sigma : (\tilde{X}, E) \to (X, x)$ be the minimal resolution of the germ of the quotient surface singularity (X, x).

$$\hbar^0(x,m) := \dim[H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}})/H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}})] \tag{4}$$

$$h^1(x,m) := h^0(X, R^1\sigma_* S^m\Omega^1_{\tilde{X}}) \tag{5}$$

From local cohomology on the germ (X, x) of a canonical singularity, we have the following inequality (which appears in Theorem 2(1)) between the invariants $\hbar^0(x,m)$ and $h^1(x,m)$: Let (X, x) be the germ of a canonical surface singularity; then, for all $m \geq 0$

$$\hbar^0(x,m) \leq h^1(x,m).$$

Proof of Theorem 2(1). Let X be an affine surface with a single canonical singularity x and $\sigma : \tilde{X} \to X$ a minimal resolution with exceptional set E. The long exact sequence for local cohomology:

$$0 \to H^0_E(\tilde{X}, S^m\Omega^1_{\tilde{X}}) \to H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}}) \to H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}}) \to H^1_E(\tilde{X}, S^m\Omega^1_{\tilde{X}}),$$

in conjunction with $H^0_E(\tilde{X}, S^m\Omega^1_{\tilde{X}}) = 0$ (since $S^m\Omega^1_{\tilde{X}}$ is torsion free) gives that:

$$\dim[H^0(\tilde{X} \setminus E, S^m\Omega^1_{\tilde{X}})/H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}})] \leq h^1_E(\tilde{X}, S^m\Omega^1_{\tilde{X}})$$

Proposition 2.3 of [17] (see also [18]) on the cohomology with compact supports on neighborhoods of exceptional sets states that

$$h^1_E(\tilde{X}, S^m\Omega^1_{\tilde{X}}) = h^1_c(\tilde{X}, S^m\Omega^1_{\tilde{X}}),$$

the subscript c indicating cohomology with compact supports.

We then use the equality

$$h^1_c(\tilde{X}, S^m\Omega^1_{\tilde{X}}) = h^1(\tilde{X}, (S^m\Omega^1_{\tilde{X}})^\vee \otimes K_{\tilde{X}})$$

that follows from a version of Serre duality (see [24, p.225]). More precisely, we have the cup product pairing

$$H^1(\tilde{X}, (S^m\Omega^1_{\tilde{X}})^\vee \otimes K_{\tilde{X}}) \times H^1_c(\tilde{X}, S^m\Omega^1_{\tilde{X}}) \to H^2_c(X, K_{\tilde{X}}),$$

and the trace map $H^2_c(\tilde{X}, K_{\tilde{X}}) \to \mathbb{C}$ gives a duality morphism:

$$\varphi : H^1_c(\tilde{X}, S^m\Omega^1_{\tilde{X}}) \to H^1(\tilde{X}, (S^m\Omega^1_{\tilde{X}})^\vee \otimes K_{\tilde{X}}),$$

which is surjective and injective if $h^1(\tilde{X}, (S^m\Omega^1_{\tilde{X}})^\vee \otimes K_{\tilde{X}})$ and $h^2(\tilde{X}, (S^m\Omega^1_{\tilde{X}})^\vee \otimes K_{\tilde{X}})$ are finite dimensional, respectively. The finiteness follows from [25] and holds for all coherent sheaves on a strictly pseudo-convex surface, and it is also a consequence of applying the Leray spectral sequence to a projective surface with such a singularity.

Combining the previous equalities with the property that a canonical surface singularity has $K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$, plus the fact the ${\Omega^1_{\tilde{X}}}^\vee \cong \Omega^1_{\tilde{X}} \otimes K^\vee_{\tilde{X}}$, we obtain that

$$h^1_E(\tilde{X}, S^m\Omega^1_{\tilde{X}}) = h^1(\tilde{X}, S^m\Omega^1_{\tilde{X}})$$

and the result follows. □

2.4 Relations between Euler characteristics of symmetric powers of the cotangent bundle on orbifold surfaces and their singularity invariants

2.4.1 The relation between $\chi(X, \hat{S}^m\Omega^1_X)$ and $\chi_{\mathrm{orb}}(X, \hat{S}^m\Omega^1_X)$

Define

$$\chi_{\mathrm{orb}}(X, \mathcal{F}) := \int_X \mathrm{ch}_{\mathrm{orb}}(\mathcal{F})\mathrm{td}_{\mathrm{orb}}(X) \tag{6}$$

to be the orbifold Euler characteristic of the orbifold vector bundle $\mathcal{F}$ (as in [11]). The orbifold Chern character $\mathrm{ch}_{\mathrm{orb}}(\mathcal{F})$ of $\mathcal{F}$ and the orbifold Todd class $\mathrm{td}_{\mathrm{orb}}(X)$ of T_X involve the orbifold Chern classes of the sheaf $\mathcal{F}$ and the tangent bundle of X (for constructions of orbifold Chern classes see [23] Section 2).

Definition 2 Let (X, x) be a germ of quotient surface singularity. Consider the invariants, $m \geq 1$:

$$\mu(x, m) := \frac{1}{|G_x|} \sum_{g \in G_x \setminus \{\mathrm{Id}\}} \frac{\mathrm{Tr}(\rho_{\hat{S}^m\Omega^1_X}(g))}{\det(\mathrm{Id} - g)} \tag{7}$$

where $G_x \subset GL(2, \mathbb{C})$ is the local fundamental group and $\rho_{\hat{S}^m\Omega^1_X}$ the representation of G_x associated to the orbifold vector bundle $\hat{S}^m\Omega^1_X$ (see [11] 2.6 for the bijective association of isomorphism classes of representations of G_x to isomorphism classes of germs of orbifold vector bundles at the quotient singularity with local fundamental group G_x).

If X is smooth and the sheaf $\mathcal{F}$ is locally free, then the Hirzebruch-Rieman-Roch states that $\chi(X, \mathcal{F}) = \chi_{\mathrm{orb}}(X, \mathcal{F})$. In the case X is an orbifold with isolated singularities and $\mathcal{F}$ is an orbifold vector bundle, the Euler characteristics $\chi(X, \mathcal{F})$ and $\chi_{\mathrm{orb}}(X, \mathcal{F})$ no longer coincide. The difference is concentrated on the singularities.

Proposition 8 *Let X be a compact orbifold surface. Then*

$$\chi(X, \hat{S}^m\Omega^1_X) = \chi_{\mathrm{orb}}(X, \hat{S}^m\Omega^1_X) + \sum_{x \in Sing(X)} \mu(x, m) \tag{8}$$

Proof Each germ of a quotient singularity (X, x) has uniquely defined group homomorphisms $\mu(x, -) : K_{\mathrm{orb}}((X, x)) \to \mathbb{Q}$, where $K_{\mathrm{orb}}((X, x))$ is the Grothendieck group of orbifold vector bundles on (X, x), such that for a compact orbifold surface X and orbifold vector bundle $\mathcal{F}$ the following holds:

$$\chi(X, \mathcal{F}) = \chi_{\mathrm{orb}}(X, \mathcal{F}) + \sum_{x \in \mathrm{Sing}(X)} \mu(x, [\mathcal{F}]_x),$$

where $[\mathcal{F}]_x$ is the class of $\mathcal{F}$ in $K_{\mathrm{orb}}((X, x))$ ([12, §3] and [11, §3.5]).

The representation-theoretic expression (7) for the invariants $\mu(x, m)$ is derived from the Atiyah-Singer equivariant Riemann-Roch (see for example [11, 3.17]) applied to the case $\mu(x, [\hat{S}^m\Omega^1_X]_x)$. □

2.4.2 Local Chern numbers and the relation between $\chi_{\mathrm{orb}}(\tilde{X}, S^m\Omega^1_{\tilde{X}})$ and $\chi_{\mathrm{orb}}(X, \hat{S}^m\Omega^1_X)$

Let $\sigma : (\tilde{X}, E) \to (X, x)$ be the minimal resolution of the germ of a quotient surface singularity. Associated to a locally free sheaf $\mathcal{F}$ on $\tilde{X}$, one has the relative local Chern classes $c_i(x, \mathcal{F}) \in$

$H^{2i}(\tilde{X}, \partial X, \mathbb{Q})$, with $H^4(\tilde{X}, \partial X, \mathbb{Q}) = \mathbb{Q}$ (cf. [10, §0] and [12, §2]). Alternatively, the local Chern classes

$$c_i(x, \mathcal{F}) \in H^{2i}_{\mathrm{dRc}}\left((\tilde{X}, E), \mathbb{Q}\right),$$

where dRc stands for de Rham cohomology with compact supports (cf. [11, §3]).

From the local Chern classes we can obtain local Chern numbers. We are interested in the following:

$$c_1^2(x) := c_1^2(x, T_{\tilde{X}}) = c_1^2(x, \Omega^1_{\tilde{X}}) \in \mathbb{Q},$$
$$c_2(x) := c_2(x, T_{\tilde{X}}) = c_2(x, \Omega^1_{\tilde{X}}) \in \mathbb{Q}$$
$$s_2(x) = c_1^2(x) - c_2(x)$$

The following facts will be important later:

i)
$$c_2(x) = e(E) - \frac{1}{|G_x|}, \tag{9}$$

with $e(E)$ the topological Euler characteristic of the exceptional locus and $|G_x|$ the order of the local fundamental group ([11] 3.18).

ii) If (X, x) is the germ of a canonical surface singularity, then

$$s_2(x) = -c_2(x), \tag{10}$$

since $c_1^2(x) = 0$, as the canonical divisor is trivial in a neighborhood of the exceptional set of the minimal resolution of a canonical singularity.

Definition 3 Let (X, x) be a germ of a quotient surface singularity. The local orbifold Euler characteristics is given by

$$\chi_{\mathrm{orb}}(x, m) := \frac{s_2(x)}{3!} m^3 - \frac{1}{2} c_2(x) m^2 - \frac{c_1^2(x) + 3c_2(x)}{12} m + \frac{c_1^2(x) + c_2(x)}{12} \tag{11}$$

Let X be a compact orbifold surface and $\tilde{X}$ its minimal resolution. The relation between the orbifold Euler characteristics of $S^m\Omega^1_{\tilde{X}}$ on $\tilde{X}$ and of $\hat{S}^m\Omega^1_X$ on X involves the local orbifold Euler characteristics at the singularities.

Proposition 9 *Let X be a compact orbifold surface, $\tilde{X} \xrightarrow{\sigma} X$ be the minimal resolution. The following holds:*

$$\chi(\tilde{X}, S^m\Omega^1_{\tilde{X}}) = \chi_{\mathrm{orb}}(\tilde{X}, S^m\Omega^1_{\tilde{X}}) = \chi_{\mathrm{orb}}(X, \hat{S}^m\Omega^1_X) + \sum_{x \in \mathrm{Sing}(X)} \chi_{\mathrm{orb}}(x, m) \tag{12}$$

Proof The first equality holds since $\tilde{X}$ is smooth. The local orbifold Euler characteristics $\chi_{\mathrm{orb}}(x, m)$ have the same topological expression as the global counterpart for $\hat{S}^m\Omega^1_X$ or $S^m\Omega^1_{\tilde{X}}$ but with the global Chern numbers replaced by the local Chern numbers. The lemma then follows from Proposition 3.14 of [11] which relates local and global Chern numbers. □

2.4.3 The relation between the singularity invariants

Proposition 10 *Let (X, x) be the germ of a surface quotient singularity. Then for all $m \geq 0$,*

$$h^1(x, m) = \mu(x, m) - \chi_{\text{orb}}(x, m) - \hbar^0(x, m) \tag{13}$$

Proof It follows from combining (2), Proposition 8, and Proposition 9. □

2.5 Asymptotics

If X is the minimal resolution of an orbifold surface Y, the asymptotics of $h^1(y, m)$ for quotient singularities $y \in \text{Sing}(Y)$ will be used in Section 2 to derive a lower bound for the m-asymptotics of $h^1(X, S^m\Omega_X^1)$.

2.5.1 The asymptotics of $\mu(x, m)$ and $\chi_{\text{orb}}(x, m)$

Proposition 11 *Let (X, x) be the germ of a surface quotient singularity. Then*

$$\lim_{m\to\infty} \frac{\mu(x, m)}{m^3} = 0 \tag{14}$$

$$\lim_{m\to\infty} \frac{\chi_{\text{orb}}(x, m)}{m^3} = -\frac{s_2(x)}{3!} \tag{15}$$

Proof Equality (15) is immediate from expression (11). Equality (14) is a particular case of the general result stating that $\lim_{m\to\infty} \frac{\mu(x, [\hat{S}^m\mathcal{F}])}{m^3} = 0$ for every rank 2 reflexive sheaf $\mathcal{F}$ on a quotient surface singularity (see [10], [11, 4.4] or [12]). □

2.5.2 The asymptotics of $h^1(x, m)$

Definition 4 The limit

$$h^1_\Omega(x) = \liminf_{m\to\infty} \frac{h^1(x, m)}{m^3} \tag{16}$$

is called the 1-*st cohomological* Ω_X*-asymptotics* of the quotient surface singularity (X, x).

Remark 1 In Section 2.6, we will see that for A_n singularities, the liminf in (16) is the limit. We expect this fact to also hold for all quotient singularities.

We can finally establish the following result (which appears in Theorem 2(2)) which plays a key role in Section 2 when comparing the bigness criteria: let (X, x) be the germ of a canonical surface singularity; then

$$h^1_\Omega(x) \geq \frac{c_2(x)}{2 \cdot 3!} \tag{17}$$

Proof of Theorem 2(2). From Propositions 2 and 3, we have the following asymptotic relation

$$\lim_{m\to\infty} \frac{\hbar^0(x, m) + h^1(x, m)}{m^3} = -\frac{s_2(x, T_{X_{\min}})}{3!}. \tag{18}$$

For canonical singularities, we have from (10) that $s_2(x) = -c_2(x)$, and the claim then follows from Proposition 1 stating that $\hbar^0(x, m) \leq h^1(x, m)$ for all $m \geq 0$. □

2.6 A_n singularities

We can find all the invariants we have considered for A_n singularities. In [16], a non-closed formula for finding $\hbar^0(A_n, m)$ (i.e. $\hbar^0(x, m)$ when x is a A_n singularity) was established. In part II of this work, we adapt the method in [16] to show that for n fixed $\hbar^0(A_n, m)$ is a quasi-polynomial in m of degree 3 (the coefficients are periodic functions of m), and more importantly, we find a closed formula in n for the leading coefficient $\hbar^0_\Omega(A_n)$. From this follows a closed formula for $h^1_\Omega(A_n)$ that allows us to establish the strength of the CMS criterion in Section 2 and 3.

We also establish the formula for $\mu(A_n, m)$. We show that for n fixed, $\mu(A_n, m)$ are quasi-polynomials of degree 1 and 0 in m, with coefficients of period $n+1$, for n odd and n even, respectively. We note that due to (11), the formula for $\chi_{\text{orb}}(A_n, m)$ is known, and due to (13) all invariants follow from $\mu(A_n, m)$, $\hbar^0(A_n, m)$ and $\chi_{\text{orb}}(A_n, m)$. An application of these non-asymptotic invariants appears in Section 4 where we analyze the minimal degrees of symmetric differentials that can occur on deformations of smooth hypersurfaces of degree d in $\mathbb{P}^3$.

We start by referencing and giving a short description of the results of Part II that are important for Sections 2 and 3. In Theorem 1 (II) (i.e. theorem 1 of part II), we show that for each n and m, $\hbar^0(A_n, m)$ is given by a weighted lattice sum over a polygon $\mathcal{P}_n(m)$,

$$\hbar^0(A_n, m) = \sum_{\substack{\mathbf{x}=(x_1,x_2)\in\mathcal{P}_n(m)\cap\mathbb{Z}^2 \\ x_1+(n+1)x_2\equiv m \bmod 2}} h_{n,m}(\mathbf{x}).$$

If we fix n, then $\hbar^0(A_n, m)$ is a quasi-polynomial in m of degree 3 with constant leading coefficient $\hbar^0_\Omega(A_n)$ for which we present a closed formula in n determined in Part (b) of the theorem. We also obtain that

$$\lim_{n\to\infty} \hbar^0_\Omega(A_n) = \frac{2\pi^2}{9} - 2 \tag{19}$$

In terms of bigness of the cotangent bundle, the relevant invariant is the 1-st cohomological Ω_X-asymptotics $h^1_\Omega(A_n)$ of A_n. Theorem 2(II) gives that $h^1_\Omega(A_n) = \lim_{m\to\infty} \frac{h^1(A_n,m)}{m^3}$ (the limit exists) and we obtain the closed formula

$$h^1_\Omega(A_n) = \frac{n^5 + 19n^4 + 83n^3 + 137n^2 + 80n}{6(n+1)^2(n+2)^2} - \frac{4}{3}\sum_{k=1}^{n}\frac{1}{k^2}. \tag{20}$$

Corollary 2 *The invariant $h^1_\Omega(A_n)$ satisfies:*

i)
$$\lim_{n\to\infty} h^1_\Omega(A_n) = \infty \tag{21}$$

ii)
$$\frac{n}{6} - \frac{4\pi^2 - 39}{18} < h^1_\Omega(A_n) < \frac{n}{6} \tag{22}$$

iii)
$$h^1_\Omega(A_{n_1+n_2}) > h^1_\Omega(A_{n_1}) + h^1_\Omega(A_{n_2}) \tag{23}$$

The comparison of the bigness criteria in later sections involves the ratio between the 1-st cohomological Ω-asymptotics and $\frac{c_2(A_n)}{2\cdot 3!}$ (the contribution from an A_n singularity in the RR-criterion, see introduction).

Corollary 3 *The ratio* $\dfrac{h^1_\Omega(A_n)}{\frac{c_2(A_n)}{2\cdot 3!}}$ *increases with n and:*

$$\lim_{n\to\infty} \frac{h^1_\Omega(A_n)}{\frac{c_2(A_n)}{2\cdot 3!}} = 2 \tag{24}$$

Proof This follows from (18), (19), (21). □

Table 2: 1-st cohomological Ω-asymptotics and its ratio to $\frac{c_2(A_n)}{2\cdot 3!}$ for low n

n	1	2	3	4	5	6	$\gg 0$
$h^1_\Omega(A_n)$	$\frac{4}{27}$	$\frac{67}{216}$	$\frac{1283}{2700}$	$\frac{577}{900}$	$\frac{106\,819}{132\,300}$	$\frac{1\,030\,727}{1\,058\,400}$	$\approx \frac{n}{6} - \frac{4\pi^2-39}{18}$
$\frac{h^1_\Omega(A_n)}{\frac{c_2(A_n)}{2\cdot 3!}}$	$\frac{32}{27}$	$\frac{67}{48}$	$\frac{5132}{3375}$	$\frac{577}{360}$	$\frac{213\,638}{128\,625} \approx 1.66$	$\frac{1\,030\,727}{604\,800} \approx 1.70$	≈ 2

Question: $h^1_\Omega(A_n) < \frac{n}{6}$ holds for $n \geq 1$. Do the inequalities $h^1_\Omega(D_n) < \frac{n}{6}$, $n \geq 4$ and $h^1_\Omega(E_m) < \frac{m}{6}$, $m = 6, 7, 8$ also hold?

The invariants $\chi_{\text{orb}}(A_n, S^m\Omega^1_{\tilde{X}})$ for an A_n singularity are given from the same topological expression as (11) but with the local Chern and Segre numbers:

$$c_1^2(A_n, \Omega^1_{\tilde{X}}) = 0 \qquad \text{and} \qquad c_2(A_n, \Omega^1_{\tilde{X}}) = \frac{n(n+2)}{n+1}$$

Hence

$$\chi_{\text{orb}}(A_n, S^m\Omega^1_{\tilde{X}}) = -\frac{n^2+2n}{6(n+1)}\left(m^3 + 3m^2 + \frac{3}{2}m - \frac{1}{2}\right) \tag{25}$$

We turn our attention to the invariants $\mu(A_n, m) := \mu(A_n, [\hat{S}^m\Omega^1_X])$. A key feature of these invariants is that for all quotient singularities $\lim_{m\to\infty} \frac{\mu(x,[\hat{S}^m\Omega^1_X])}{m^3} = 0$, as was described in Proposition 11. Hence they have no impact towards the 1-st cohomological Ω_X-asymptotics of quotient singularities.

We are interested in obtaining a formula for $\mu(A_n, m)$, the missing part in (13) to determine a formula for $h^1(x, m)$. The latter can be used to determine if a surface X with A_n singularities has symmetric differentials of degree m (see section 3.2).

Proof (of Theorem 6) It follows from (7) of section 1.7 that $\mu(A_n, m)$ can be computed via the representation theoretical formula:

$$\mu(A_n, m) = \frac{1}{|n+1|} \sum_{g \in G^*_{n+1}} \frac{\operatorname{Tr}(\rho_{\hat{S}^m\Omega^1_X}(g))}{\det(\operatorname{Id} - g)}, \tag{26}$$

where $G_{n+1} := \left\langle \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} \right\rangle$, ε is a primitive $(n+1)$-root of unity, $G^*_{n+1} = G_{n+1} \setminus \{\text{Id}\}$ and $\rho_{\hat{S}^m\Omega^1_X}$ is the representation canonically associated with the reflexive module $(\hat{S}^m\Omega^1_X)_{A_n}$ over the local ring $\mathcal{O}_{A_n}$. This representation can be obtained via the structural action of G_{n+1} on the smoothing $(\mathbb{C}^2, z_1, z_2)$ of (X, A_n); more precisely, via its induced action on the fiber of the vector bundle $(S^m\Omega^1_{\mathbb{C}^2})$ at $(0, 0)$.

The representation $\rho_{\hat{S}^m\Omega^1_X}$ is given by:

$$\rho_{\hat{S}^m\Omega^1_X}\left(\begin{bmatrix}\varepsilon^i & 0\\ 0 & \varepsilon^{-i}\end{bmatrix}\right) = \begin{bmatrix}\varepsilon^{-im} & & & \\ & \varepsilon^{-im+2i} & & \\ & & \ddots & \\ & & & \varepsilon^{im}\end{bmatrix}. \tag{27}$$

Determining (26) using (27) gives:

$$\mu(A_n, m) = \frac{1}{n+1}\sum_{k=0}^{m}\sum_{i=1}^{n}\frac{(\varepsilon^i)^{m-2k}}{2-\varepsilon^i-\varepsilon^{-i}}. \tag{28}$$

Using $f_n(j) := \sum_{i=1}^{n}\frac{(\varepsilon^i)^j}{2-\varepsilon^i-\varepsilon^{-i}}$, we re-express (28):

$$\mu(A_n, m) = \frac{1}{n+1}\sum_{k=0}^{m} f_n(m-2k), \tag{29}$$

Note that

$$f_n(j) = f_n(j') \text{ if } j \equiv j' \pmod{n+1} \tag{30}$$

The sums appearing in $f_n(j)$ are well-studied. We have Lemma 3.3.2.1 of [26], stating that if $0 \leq j \leq n$, then:

$$f_n(j) = \sum_{i=1}^{n}\frac{(\varepsilon^i)^j}{2-\varepsilon^i-\varepsilon^{-i}} = \frac{j(j-(n+1))}{2} + \frac{n^2+2n}{12} \tag{31}$$

Then, the following are immediate

$$\sum_{j=0}^{n} f_n(j) = 0 \qquad\qquad \text{for all } n \geq 0 \tag{32}$$

$$\sum_{j=0}^{\frac{n-1}{2}} f_n(2j) = \frac{n+1}{8}\ ; \qquad \sum_{j=0}^{\frac{n-1}{2}} f_n(2j+1) = -\frac{n+1}{8} \qquad \text{for all } n \geq 0 \text{ and odd} \tag{33}$$

For n even, $\mu(A_n, m)$ as a function of m attains at most $n+1$ values, so, in particular, it is bounded. To see this set:

$$\alpha_n(r) = \frac{1}{n+1}\sum_{k=0}^{r} f_n(r-2k) \tag{34}$$

If n even, then for any $i \in \mathbb{Z}$ $\{m-2i, m-2(i+1), ..., m-2(i+n)\} \equiv \{0, ..., n\} \pmod{n+1}$. Now use (30) and (32) to obtain that $\sum_{k=i}^{i+n} f_n(m-2k) = 0$ for any i. All this can be used to derive, after setting $r \equiv m \pmod{n+1}$ and $0 \leq r \leq n$, that:

$$\sum_{k=0}^{m} f_n(m-2k) = \sum_{k=0}^{m} f_n(r-2k) = \sum_{k=0}^{r} f_n(r-2k) + \sum_{k=r+1}^{m} f_n(r-2k) = \alpha_n(r), \tag{35}$$

and hence

$$\text{for } n \text{ even: } \ \mu(A_n, m) = \alpha_n(r), \ \ r \equiv m \pmod{n+1} \text{ and } 0 \leq r \leq n$$

The formula for $\alpha_n(r)$ appearing in the statement of the theorem follows from (34) (we omit the non-enlightening algebraic manipulations involved).

If n is odd, $\sum_{k=i}^{i+n} f_n(m-2k) = 0$ no longer holds, we have using (30), (34) and (33) (instead of (32)):

$$\sum_{k=0}^{m} f_n(m-2k) = \sum_{k=0}^{r} f_n(r-2k) + \sum_{k=r+1}^{m} f_n(r-2k) = \alpha_n(r) + \frac{(-1)^m}{4}\lfloor\frac{m+1}{n+1}\rfloor \tag{36}$$

Again with $r \equiv m \pmod{n+1}$ and $0 \leq r \leq n$.

□

3 Bigness criteria

The cotangent bundle Ω^1_X on a complex manifold X of dimension n is said to be big if

$$\lim_{m\to\infty} \frac{h^0(X, S^m\Omega^1_X)}{m^{2n-1}} \neq 0,$$

i.e., the symmetric pluri-genera $P^S_m(X) := h^0(X, S^m\Omega^1_X)$ has the maximal growth order possible with respect to m for $\dim X = n$. The symmetric plurigenera are birational invariants and hence the cotangent bundle being big is a birational property (among smooth representatives).

From now on, we restrict our attention to the surface case. Moreover, we are only interested in surfaces X of general type since Ω^1_X being big implies the canonical bundle K_X is also big, see for example [1].

3.1 Basic asymptotics of the symmetric plurigenera

The ingredients in the asymptotic behaviour of $h^0(X, S^m\Omega^1_X)$ are determined by Riemann-Roch and Bogomolov's vanishing

$$H^2(X, S^m\Omega^1_X) = 0, \qquad m \geq 3.$$

This follows from applying Serre duality to Bogomolov's vanishing theorem for surfaces of general type:

$$H^0(X, S^m T_X \otimes K^p_X) = 0, \qquad m - 2p > 0$$

The latter vanishing follows from the K_X-semi-stability properties of T_X for minimal surfaces of general type [7] and [8] (see Lemma 1). Hence for $m \geq 3$:

$$h^0(X, S^m\Omega^1_X) = h^1(X, S^m\Omega^1_X) + \int_X \mathrm{ch}(S^m\Omega^1_X)\,\mathrm{td}(X) \tag{37}$$

Hence, the asymptotics of $h^0(X, S^m\Omega^1_X)$ is given by

$$\lim_{m\to\infty} \frac{h^0(X, S^m\Omega^1_X)}{m^3} = \lim_{m\to\infty} \frac{h^1(X, S^m\Omega^1_X)}{m^3} + \frac{s_2(X)}{3!}. \tag{38}$$

Note that in (38), the limits are regular limits, and there is no need to use limsup since

(1) $h^0(X, S^m\Omega^1_X) = h^0(\mathbb{P}(\Omega^1_X), \mathcal{O}(m))$ and the volume of a line bundle on a projective variety (in this case $\mathcal{O}(1)$ on $\mathbb{P}(\Omega^1_X)$) being given by a regular limit [27, 11.4.A], and

(2) the vanishing of $h^2(X, S^m\Omega^1_X)$ for $m \geq 3$ and the polynomial expression for $\int_X \mathrm{ch}(S^m\Omega^1_X)\,\mathrm{td}(X)$:

$$\int_X \mathrm{ch}(S^m\Omega^1_X)\,\mathrm{td}(X) = \frac{s_2}{3!}m^3 - \frac{1}{2}c_2 m^2 - \frac{1}{12}(c_1^2 + 3c_2)m + \frac{1}{12}(c_1^2 + c_2)$$

with $c_1^2 := c_1^2(X)$ and $c_2 := c_2(X)$ the Chern numbers of X and $s_2 = c_1^2 - c_2$ the 2^{nd} Segre number of X.

3.2 Birational classes and invariants

The bigness of the cotangent bundle being a birational property motivated us to express part of the bigness criterion in a birational form.

We consider birational classes $\mathcal{X}$ of surfaces of general type, i.e. containing smooth surfaces of general type. Any such class $\mathcal{X}$ has two special representatives

- $X_{\min}$, the minimal model of $\mathcal{X}$ (smooth and $K_{X_{\min}}$ nef), and
- X_{can}, the canonical model of $\mathcal{X}$ (canonical surface singularities and $K_{X_{\text{can}}}$ ample).

For any smooth $X_1 \in \mathcal{X}$, we have

$$X_1 \xrightarrow{\varphi} X_{\min} \xrightarrow{\varphi_c} X_{\text{can}},$$

where φ is a composition of blow downs of (-1)-curves (rational curves E with $E \cdot E = -1$) and φ_c a contraction of ADE (-2)-curves configurations; recall that a (-2)-curve is a rational curve E with $E \cdot E = -2$). Note that $X_{\min}$ is, in fact, the minimal resolution of X_{can}.

Definition 5 If a birational class $\mathcal{X}$ has smooth representatives with big cotangent bundle, then we say that $\mathcal{X}$ has big cotangent bundle $\Omega^1_{\mathcal{X}}$.

There are two birational invariants that are involved in determining bigness of the cotangent bundle.

Definition 6 Let $\mathcal{X}$ be a birational class of surfaces of general type.

i) the 2^{nd} Segre number of $\mathcal{X}$ is $s_2(\mathcal{X}) := s_2(X_{\min})$.

ii) the 1^{st}-cohomological Ω-asymptotics of $\mathcal{X}$ is:

$$h^1_\Omega(\mathcal{X}) := \lim_{m\to\infty} \frac{h^1(X_{\min}, S^m\Omega^1_{X_{\min}})}{m^3}. \tag{39}$$

If X is a smooth surface of general type, the 1st-cohomological Ω-asymptotics of X is also defined by

$$h^1_\Omega(X) := \lim_{m\to\infty} \frac{h^1(X, S^m\Omega^1_X)}{m^3}.$$

We note that $s_2(\mathcal{X})$ and $h^1_\Omega(\mathcal{X})$ are respectively the maximum of the $s_2(X)$ and the minimum of the $h^1_\Omega(X)$ for all smooth representatives X of $\mathcal{X}$.

3.3 Birational version of the topological criterion for bigness

The simplest and most commonly used criterion for the bigness of the cotangent bundle of a surface of general type follows from (2.2),

Criterion 1. (Topological bigness criterion) *A smooth surface X of general type has big Ω^1_X if $s_2(X) > 0$.*

This criterion is not birational since blowing up a point on a surface X decreases the $s_2(X)$ by 2.

Criterion 1a. (Birational version of the topological bigness criterion) A birational class $\mathcal{X}$ of surfaces of general type has big $\Omega^1_{\mathcal{X}}$ if $s_2(\mathcal{X}) > 0$.

There are few known examples of birational classes $\mathcal{X}$ of surfaces with big cotangent bundle that fail the birational version of the topological bigness criterion. One can find them in [9] and [19] for example. To understand these examples, one needs to consider information on the canonical model of $\mathcal{X}$; we will in particular focus on its singularities, to obtain bigness of the cotangent bundle.

3.4 Complete criterion for bigness

This is an immediate consequence of (38):

Criterion 2. (Complete Bigness Criterion) *A smooth surface X of general type has big Ω^1_X if and only if*

$$h^1_\Omega(X) + \frac{s_2(X)}{3!} > 0. \tag{40}$$

Note that the quantity $h^1_\Omega(X) + \frac{s_2(X)}{3!}$ is a birational invariant.

Criterion 2a. (Birational Complete Bigness Criterion) *A birational class $\mathcal{X}$ of surfaces of general type has big $\Omega^1_{\mathcal{X}}$ if and only if*

$$h^1_\Omega(\mathcal{X}) + \frac{s_2(\mathcal{X})}{3!} > 0$$

The Complete Bigness Criterion requires a handle on $h^1_\Omega(\mathcal{X})$ (or $h^1_\Omega(X)$) which typically is more delicate and illusive than the 2nd Segre number used in the topological criterion.

3.5 The QS-Bigness Criterion

In this section we get a handle on the 1-st cohomological Ω asymptotics $h^1_\Omega(X)$ where X is a surface of general type which is the minimal resolution of a normal surface Y with only quotient singularities. We need Bogomolov's vanishing result for orbifold surfaces of general type.

Lemma 1 *Let Y be a surface of general type with only quotient singularities, then $H^2(Y, \hat{S}^m\Omega^1_Y) = 0$ for $m \geq 3$.*

Proof Serre duality gives:

$$h^2(Y, \hat{S}^m\Omega^1_Y) = h^0\left(Y, (\hat{S}^m T_Y \otimes K_Y)^{\vee\vee}\right)$$

Moreover, since $(\hat{S}^m T_Y \otimes K_Y)^{\vee\vee} \cong i_*(S^m T_Y \otimes K_Y)$, where $i : Y_{reg} \hookrightarrow Y$ is the natural inclusion, it follows that:

$$h^2(Y, \hat{S}^m\Omega^1_Y) = h^0(Y_{\text{reg}}, S^m T_Y \otimes K_Y)$$

Run the minimal model program for surfaces with quotient (log terminal) singularities, see [28] or for details [29] section 10 (see also [30]). The contraction theorem, [29] 10.3, applied repeatedly to Y and its contractions give the birational morphism:

$$\phi_m : Y \to Y_m$$

where Y_m is a normal projective surface with only quotient singularities projective with K_{Y_m} nef and big.

Since K_{Y_m} is nef and big, one has $K^2_{Y_m} > 0$. If K_{Y_m} is not ample, then by Nakai-Moizeshon ampleness criterion there is a curve C in Y_m such that $K_{Y_m}.C = 0$ and hence by Hodge index theorem $C^2 < 0$. Artin's contraction criterion gives that one can contract C and obtain a normal projective surface Y'_m, the singularities of Y'_m are still only quotient singularities (see [29] 10.3). If the new surface Y'_m does not have ample canonical divisor find a curve C' as above and contract it. Apply this process repeatedly till one obtains a surface Y_c with ample K_{Y_c} (this process must terminate due to finitness of the rank of the Picard group). Hence, one has a birational morphism:

$$\phi_c : Y \to Y_c$$

where Y_c is a normal surface with only quotient singularities and K_{Y_c} ample.

The existence of Kahler-Einstein metrics on Y_c (see [13] theorem A and [14]) imply that T_{Y_c} is K_{Y_c}-semistable. Let $S_c \subset Y_c$ be the finite collection of points consisting of all the singularities of Y_c and the images of all curves contracted by the morphism ϕ_c. Set $U_c = Y_c \setminus S_c$ and $U = \phi_c^{-1}(U_c)$ in Y, U_c and U are biholomorphic. By Mehta-Ramanathan restriction theorem, see [31], for a very general curve C in the linear system $|lK_{Y_c}|$ with $l \gg 0$, $C \subset U_c$, one has:

(i) $(S^m T_{Y_{c,\mathrm{reg}}} \otimes K_{Y_{c,\mathrm{reg}}})|_C$ is semi-stable

(ii) $\deg(S^m T_{Y_{c,\mathrm{reg}}} \otimes K_{Y_{c,\mathrm{reg}}})|_C = \dfrac{(2-m)(m+1)l}{2} K_{Y_c}^2 < 0$ for $m \geq 3$.

If $h^2(Y, \hat{S}^m \Omega_Y^1) = h^0(Y_{\mathrm{reg}}, S^m T_Y \otimes K_Y) \neq 0$, then for C as above very general would have $h^0(C, (S^m T_{Y_{c,\mathrm{reg}}} \otimes K_{Y_{c,\mathrm{reg}}})|_C) \neq 0$ and (i) and (ii) produce a contradiction. □

Proof (of Theorem 1) The Leray-Serre spectral sequence for the sheaf $S^m\Omega_X^1$ and the minimal resolution morphism $\sigma : X \to Y$ combined with the vanishing $h^2(X, S^m\Omega_X^1) = 0$ for $m \geq 3$, gives for $m \geq 3$:

$$h^1(X, S^m\Omega_X^1) = h^1(Y, \sigma_* S^m \Omega_X^1) + \sum_{y \in \mathrm{Sing}(Y)} h^1(y, m) - h^2(Y, \sigma_* S^m \Omega_X^1) \tag{41}$$

where as before $h^1(y, m) := h^0(U_y, R^1\sigma_* S^m \Omega^1_{\tilde{U}_y})$ with $\sigma : \tilde{U}_y \to U_y$ the minimal resolution of an affine neighborhood of y such that $U_y \cap \mathrm{Sing}(Y) = \{y\}$. The localized component $\sum_{y \in \mathrm{Sing}(Y)} h^1(y, m)$ of $h^1(X, S^m\Omega_X^1)$ comes from the sheaf $R^1\sigma_* S^m \Omega_X^1$ supported on the singular points of Y and hence:

$$H^0\left(Y, R^1\sigma_* S^m\Omega_X^1\right) \simeq \bigoplus_{y \in \mathrm{Sing}(Y)} H^0\left(U_y, \sigma_* R^1 S^m \Omega^1_{\tilde{U}_y}\right)$$

giving

$$h^0\left(Y, R^1\sigma_* S^m\Omega_X^1\right) = \sum_{y \in \mathrm{Sing}(Y)} h^1(y, m)$$

We proceed to show $h^2(Y, \sigma_* S^m \Omega_X^1) = 0$, for $m \geq 3$. Consider:

$$0 \to \sigma_* S^m \Omega_X^1 \to \hat{S}^m \Omega_Y^1 \to Q_m \to 0$$

Left injectivity holds since $\sigma_* S^m \Omega_X^1$ is torsion free. The support of $Q_m = (\sigma_* S^m \Omega_X^1)^{\vee\vee} \big/ \sigma_* S^m \Omega_X^1$ is $\mathrm{Sing}(Y)$ (recall that $(\sigma_* S^m \Omega_X^1)^{\vee\vee} \cong \hat{S}^m \Omega_Y^1$). Hence it follows from the cohomology long exact sequence that:

$$h^2(Y, \sigma_* S^m \Omega_X^1) = h^2(Y, \hat{S}^m \Omega_X^1)$$

Applying Bogomolov's vanishing for orbifolds surfaces of general type, Lemma 1, we have that

$$h^1(X, S^m\Omega_X^1) = h^1(Y, \sigma_* S^m \Omega_X^1) + \sum_{y \in \mathrm{Sing}(Y)} h^1(y, m) \tag{42}$$

A priori, we do not know if the separate limits $\lim_{m\to\infty} \dfrac{h^1(Y, \sigma_* S^m \Omega_X^1)}{m^3}$ and $\lim_{m\to\infty} \dfrac{\sum_{y \in \mathrm{Sing}(Y)} h^1(y, m)}{m^3}$ exist. On the other hand one clearly has that

$$\sum_{y \in \mathrm{Sing}(Y)} h^1_\Omega(y) \leq \lim_{m\to\infty} \frac{h^1(X, S^m \Omega_X^1)}{m^3} \tag{43}$$

and hence the result follows from the Complete Bigness Criterion of 2.4. □

3.6 CMS-bigness criterion and the components of $h^1_\Omega(\mathcal{X})$

There is a natural application of the QS-Bigness Criterion to any minimal surface of general type X (with Y in the criterion being the canonical model of X). We obtain a criterion of bigness of the cotangent bundle for birational classes of surfaces of general type.

CMS Bigness Criterion. Let $\mathcal{X}$ be a birational class of surfaces of general type. Then the cotangent bundle $\Omega^1_{\mathcal{X}}$ is big if

$$\sum_{x \in \mathrm{Sing}(X_{\mathrm{can}})} h^1_\Omega(x) + \frac{s_2(\mathcal{X})}{3!} > 0 \tag{44}$$

The CMS Bigness Criterion is the birational criterion that deals with the components of the Complete bigness criterion (40) that are naturally easier to get a handle on; that is the criterion is concerned only with topological data of the minimal model plus the local information encoded in the singularities of the canonical model.

The analytical component, $h^1_\Omega(X)$, of the Complete bigness criterion is the delicate component of this criterion. For example $h^1_\Omega(X)$ can change under smooth deformations (see next section), while the topological component $\frac{s_2(X)}{3!}$ is preserved under deformation. The CMS-bigness criterion deals with the localized component of $h^1_\Omega(X)$ which is derived from the sheaves $R^1\sigma_* S^m\Omega^1_{X_{\mathrm{min}}}$, $m \geq 0$, supported at the singular points of X_{can}, $\sigma : X_{\mathrm{min}} \to X_{\mathrm{can}}$.

Definition 7 Let $\mathcal{X}$ be a birational class of surfaces of general type with minimal model X_{min} and canonical model X_{can}. We define the localized component of the 1-st cohomological $\Omega^1_{\mathcal{X}}$ asymptotics, $h^1_\Omega(\mathcal{X})$, to be:

$$Lh^1_\Omega(\mathcal{X}) := \sum_{x \in \mathrm{Sing}(X_{\mathrm{can}})} h^1_\Omega(x) \tag{45}$$

and the nonlocalized component of $h^1_\Omega(\mathcal{X})$ to be:

$$NLh^1_\Omega(\mathcal{X}) := \liminf_{m \to \infty} \frac{h^1(X_{\mathrm{can}}, \sigma_* S^m \Omega^1_{X_{\mathrm{min}}})}{m^3}. \tag{46}$$

We can also define the localized and nonlocalized components of $h^1_\Omega(X)$ for any surface of general type X. If X is minimal then the expressions are verbatim those in (45) and (46). If the surface X is not minimal, then the localized component also would have contributions coming from each of the blow ups required to obtain X from X_{min}.

Proposition 12 *Let $\mathcal{X}$ be a birational class of surfaces of general type whose canonical model has only singularities of type A_n. Then:*

$$h^1_\Omega(\mathcal{X}) = Lh^1_\Omega(\mathcal{X}) + NLh^1_\Omega(\mathcal{X}) \tag{47}$$

Proof The identity (47) follows from (42) in the proof of Theorem 1 applied to the minimal model of $\mathcal{X}$ plus the existence of the limit $\lim_{m\to\infty} \frac{h^1(A_n,m)}{m^3}$ for all $n \geq 1$ that is guaranteed in Theorem 2(II). □

Proposition 12 should hold in the general case, i.e., when we do not restrict the canonical model of $\mathcal{X}$ to have only A_n singularities.

This article is primarily focused on the localized component of $h^1_\Omega(\mathcal{X})$. Future work will be dedicated to getting a handle on the non-localized component of $h^1_\Omega(\mathcal{X})$.

3.7 CMS criterion range

We consider the question about which pairs of Chern numbers (c_2, c_1^2) cannot be attained by a birational class $\mathcal{X}$ satisfying the CMS-criterion and address the implications towards the geography of surfaces with big cotangent bundle.

Let $\mathcal{X}$ be a surface birational class of general type, its pair of Chern numbers $(c_2(\mathcal{X}), c_1^2(\mathcal{X}))$ are defined to be $(c_2(X_{\min}), c_1^2(X_{\min}))$ and for notation simplicity denoted by (c_2, c_1^2). More generally, we set all the topological invariants of a birational class to be those of its minimal model.

Let GT, CGT and the excluded range for a criterion C, ER(C)$\subset CGT$, be as in the introduction. It is known that

$$CGT \subset \{(c_2, c_1^2) \mid \frac{1}{5}(c_2 - 36) \leq c_1^2 \leq 3c_2,\ c_2 + c_1^2 \equiv_{12} 0\},$$

the limiting lines follow from the Bogomolov-Miyaoka-Yau (above) and Noether's (below) inequalities, the modular condition follows from Noether's formula $c_2 + c_1^2 = 12\chi$, see [32].

It is straightforward that all the pairs of Chern numbers lying in the region $\{c_1^2 > c_2\} \cap CGT$, are attained by $\mathcal{X}$ satisfying the CMS-criterion, due to the topological bigness criterion. However, the CMS-criterion is not topological, in the sense that the topology of $\mathcal{X}$ might not determine whether the criterion condition, $Lh^1_\Omega(\mathcal{X}) + \frac{s_2(\mathcal{X})}{3!} > 0$, holds or not. Examples illustrating this appeared in [9], see Section 3.1.

We turn towards investigating the region ER(CMS)$\subset \{c_1^2 \leq c_2\} \cap CGT$, i.e. the region the CMS-criterion cannot hold. The tools to address this problem must be results where the Chern numbers of $\mathcal{X}$ bound the canonical singularities of the canonical model. To our knowledge, there are two such general results as mentioned in the introduction: the H-bound (**) and the M-bound (*); the bound coming from standard Hodge theory (see below) and the Miyaoka bound [21], respectively.

The H-bound constrains the singularities of the canonical model by bounding

$$\rho_{-2}(\mathcal{X}) := \#\{(-2)\text{-curves on } X_{\min}\}$$

The standard reasoning behind the H-bound is: given a smooth projective surface Y, then $\rho_{-2}(Y) \leq \rho(Y) - 1 \leq h^{1,1}(Y) - 1$, with $\rho(Y)$ the Picard number of Y; the homology classes of the $(-2)-$curves are linearly independent in $H_2(Y, \mathbb{C})$ (the connected configurations of $(-2)-$curves have negative definite intersection pairing); then using $b_2(Y) = h^{1,1}(Y) + 2p_g(Y)$, $c_2(Y) = 2 - 2b_1(Y) + b_2(Y)$ and $12(1 - \frac{1}{2}b_1(Y) + p_g(Y)) = c_1^2(Y) + c_2(Y)$, we obtain:

$$\text{(H-bound)} \quad \rho_{-2}(\mathcal{X}) \leq \frac{5c_2(\mathcal{X}) - c_1^2(\mathcal{X})}{6} + b_1(\mathcal{X}) - 1 \tag{48}$$

The M-bound is a consequence of the logarithmic analogue of the Bogomolov-Miyaoka-Yau inequality concerning minimal resolutions of normal surfaces with quotient singularities, [21, Theorem 1.1]. The M-bound corresponds to the special case concerning the pair $(X_{\min}, X_{\text{can}})$:

$$\sum_{x \in \text{Sing}(X_{\text{can}})} c_2(x) \leq c_2(\mathcal{X}) - \frac{1}{3}c_1^2(\mathcal{X}) \tag{49}$$

Remark 2 (Excluded range for the RR-criterion) As mentioned in the introduction, the RR-criterion for bigness of $\Omega^1_{\mathcal{X}}$ was formulated in [19]. We translate the RR-criterion to the framework used in this work (see Section 1.4.7): $\Omega^1_{\mathcal{X}}$ is big, if $s_{2,\text{orb}}(X_{\text{can}}) + s_2(\mathcal{X}) > 0$, or equivalently $\sum_{x \in \text{Sing}(X_{\text{can}})} \frac{c_2(x)}{2 \cdot 3!} + \frac{s_2(\mathcal{X})}{3!} > 0$. The

M-bound gives a direct bound to the singularity contribution in the criterion and it is straightforward to see that all Chern number pairs satisfying $\frac{c_1^2}{c_2} \leq \frac{3}{5}$ belong to the excluded range $ER(\mathrm{RR})$.

The difference between the CMS and the RR criteria comes from the singularity contribution to a lower bound of $h^1_\Omega(\mathcal{X})$, that is, $\sum_{x\in \mathrm{Sing}(X_{\mathrm{can}})} h^1_\Omega(x)$ versus $\sum_{x\in \mathrm{Sing}(X_{\mathrm{can}})} \frac{c_2(x)}{2\cdot 3!}$. In this respect, we have obtained two results: 1) Theorem 2(b) states that for all canonical singularities:

$$h^1_\Omega(x) \geq \frac{c_2(x)}{2\cdot 3!} \tag{50}$$

showing that CMS criterion holds whenever the RR-criterion holds; and 2) for A_n singularities,

$$h^1_\Omega(A_n) = \frac{n^5+19n^4+83n^3+137n^2+80n}{6(n+1)^2(n+2)^2} - \frac{4}{3}\sum_{k=1}^{n}\frac{1}{k^2} > \frac{n(n+2)}{12(n+1)} = \frac{c_2(A_n)}{2\cdot 3!} \tag{51}$$

In fact, the ratio $h^1_\Omega(A_n)$ to $\frac{c_2(A_n)}{2\cdot 3!}$ grows with n, with $32/27$ for $n=1$ and approaching 2 as $n\to\infty$. Therefore, if the canonical model of a birational class $\mathcal{X}$ has only singularities of type A, then the CMS criterion needs between $\frac{27}{32}$ to $\frac{1}{2}$ of the number of singularities needed by the RR criterion to guarantee bigness of the cotangent bundle.

The interaction between the M-bound and the CMS criterion is not as straightforward as it is for the RR-criterion. The contributions from each singularity, $h^1_\Omega(x)$, do not satisfy $\frac{h^1_\Omega(x)}{c_2(x)} = C$ with C independent of the type of the canonical singularity x. We start by determining how the M-bound constrains $\rho_{-2}(\mathcal{X})$.

Lemma 2 *Let $\rho_{-2,M}(c_2,c_1^2)$ denote the maximum of (-2)-curves allowed by the M-bound on $\mathcal{X}\in GT$ with pair of Chern numbers (c_2,c_1^2). Then*

$$\rho_{-2,M}(c_2,c_1^2) = \begin{cases} 0 & c_2-\frac{c_1^2}{3}=0 \\ 2 & c_2-\frac{c_1^2}{3}=\frac{8}{3} \\ \lfloor c_2-\frac{c_1^2}{3}\rfloor -1 & \textit{otherwise.}\end{cases} \tag{52}$$

Moreover, $\rho_{-2,M}(c_2,c_1^2)$ is attained by $\mathcal{X}$ having X_{can} with a single singularity of type A.

Proof Let $\rho_{-2}(x)$ be the number of (-2)-curves on the exceptional locus of the minimal resolution of a canonical singularity x and $r_{-2}(x) := \frac{c_2(x)}{\rho_{-2}(x)}$. Then,

$$r_{-2}(A_n) = \frac{n+1}{n} - \frac{1}{n(n+1)}, \qquad r_{-2}(D_n) = \frac{n+1}{n} - \frac{1}{4n(n-2)}$$

$r_{-2}(E_6) = \frac{7}{6} - \frac{1}{6\cdot 24}$, $r_{-2}(E_7) = \frac{8}{7} - \frac{1}{7\cdot 48}$ and $r_{-2}(E_8) = \frac{9}{8} - \frac{1}{8\cdot 120}$.

It follows that $r_{-2}(A_n) < r_{-2}(D_n) < r_{-2}(E_n)$ (whenever it makes sense) and $r_{-2}(A_n)$ decreases with n. Hence, the M-bound allows the most (-2)-curves if they originate from a single singularity of type A_n with highest n allowed. The result then follows from the formula for the A_n case and the fact that $c_2 - \frac{c_1^2}{3}$ must be a multiple of $\frac{4}{3}$ due to $c_2 + c_1^2 \equiv 0 \bmod 12$. □

Remark 3 The M-bound can at most force a finite collection of pairs of Chern numbers to be in the excluded range $ER(\mathrm{CMS})$. Let $(c_2,c_1^2)\in ER(\mathrm{CMS})$, then $c_1^2 \leq c_2$ which in turn implies

$$\rho_{-2,M}(c_2,c_1^2) = \lfloor c_2 - \frac{c_1^2}{3}\rfloor - 1.$$

As observed in Lemma 2, the M-bound allows the existence of an $\mathcal{X} \in GT$ having the pair of Chern numbers (c_2, c_1^2) and whose X_{can} has one A_n with $n = \lfloor c_2 - \frac{c_1^2}{3} \rfloor - 1$. Then, using $h^1_\Omega(A_n) > \frac{n}{6} - \frac{4\pi^2-39}{18}$, it follows that such $\mathcal{X}$ has $Lh^1_\Omega(\mathcal{X}) + \frac{s_2(\mathcal{X})}{3!} > 0$ unless $c_1^2 = 1, 2$. Hence, the M-bound can at most force 7 pairs to belong to $ER(\text{CMS})$, while it forces infinitely many to be in $ER(\text{RR})$.

We will see below that the H-bound imposes stronger restrictions on the CMS-criterion than the M-bound.

Lemma 3 *Let $\mathcal{X} \in GT$ be regular with a pair of Chern numbers (c_2, c_1^2). Then, the H-bound for $\rho_{-2}(\mathcal{X})$ is less than $\rho_{-2,M}(c_2, c_1^2)$ when $c_1^2 \leq c_2$.*

Proof The claim follows since in the region $c_1^2 \leq c_2$, we have $\rho_{-2,M}(c_2, c_1^2) = \lfloor c_2 - \frac{c_1^2}{3} \rfloor - 1$ while the H-bound for (a regular) $\mathcal{X}$ gives $\rho_{-2}(\mathcal{X}) < \max\{\lfloor \frac{1}{6}(5c_2 - c_1^2) \rfloor - 1, 0\}$. □

Lemma 4 *Let $\mathcal{X} \in GT$ be such that its canonical model X_{can} only has singularities of type A. Then,*

$$\frac{\rho_{-2}(\mathcal{X})}{6} - \frac{\#\,\text{Sing}(X_{\text{can}})}{37} < Lh^1_\Omega(\mathcal{X}) < \frac{\rho_{-2}(\mathcal{X})}{6}.$$

Proof Let $\text{Sing}(X_{\text{can}}) = \{x_1, ..., x_k\}$ with x_i an A_{n_i} singularity; then, $Lh^1_\Omega(\mathcal{X}) = \sum_{i=1}^k h^1_\Omega(A_{n_i})$. Since $h^1_\Omega(A_n)$ as a function of n satisfies $h^1_\Omega(A_{r+s}) > h^1_\Omega(A_r) + h^1_\Omega(A_s)$, and $h^1_\Omega(A_n) < \frac{n}{6}$ (Corollary 2), and $\sum_{i=1}^k n_i = \rho_{-2}(\mathcal{X})$, it follows that

$$\sum_{i=1}^k h^1_\Omega(A_{n_i}) \leq h^1_\Omega(A_{\rho_{-2}(\mathcal{X})}) < \frac{\rho_{-2}(\mathcal{X})}{6}.$$

For the lower bound, use $\frac{n_i}{6} - \frac{1}{37} < h^1_\Omega(A_{n_i})$ which follows from Corollary 2 and

$$\frac{1}{38} < \frac{4\pi^2 - 39}{18} < \frac{1}{37},$$

to get the claimed inequality. □

Now we are ready to prove Theorem 3.

Proof (of Theorem 3) Let $\mathcal{X} \in GT$ be such that its canonical model has only singularities of type A.

Case $c_1^2 < \frac{1}{5}c_2$:

Recall that if a minimal surface of general type Y satisfies $c_1^2(Y) < \frac{1}{5}c_2(Y)$, then Y is regular ($b_1(Y) = 0$); if $b_1(Y) \neq 0$, then Y would have unramified covers Y_d of any degree d and these covers are minimal with $c_1^2(Y_d) = dc_1^2(Y)$ and $c_2(Y_d) = dc_2(Y)$ (same ratio $\frac{c_1^2}{c_2}$ as for Y) making Y_d for d sufficiently large below the Noether's line.

Since $\mathcal{X}$ is regular, it follows from Lemma 3 and Lemma 4 that

$$Lh^1_\Omega(\mathcal{X}) < \frac{1}{36}(5c_2 - c_1^2 - 6),$$

and

$$Lh^1_\Omega(\mathcal{X}) + \frac{s_2(\mathcal{X})}{3!} < \frac{1}{36}(5c_1^2 - c_2 - 6). \tag{53}$$

This implies that any regular $\mathcal{X}$ with canonical model X_{can} with only singularities of type A will not satisfy the CMS-criterion if $c_1^2 \leq \frac{1}{5}(c_2 + 6)$ and hence this case is settled.

Case $c_1^2 = \frac{1}{5}c_2$:

The subcase where $\mathcal{X}$ is regular follows from the paragraph above.

Otherwise, Horikawa [33] (Theorem 5.1) showed that $\mathcal{X}$ has $b_1(\mathcal{X}) = 2$ and $X_{\min}$ is a fibration $f : X_{\min} \to C$ over an elliptic curve C with general fiber a curve of genus 2.

We claim that $\rho_{-2}(\mathcal{X}) \leq \rho(X_{\min}) - 2$. The claim follows since the classes of the (-2)-curves, of an hyperplane section H and of a fiber F in $H^{1,1}(X_{\min})$ are linearly independent. The (-2)-curves are necessarily contained in the fibers of f, hence the class of H can not be spanned by the classes of the (-2)-curves and of F (otherwise, $H \cdot F = 0$). Finally F can not be in the span of the classes of the $(-2)-$curves, since otherwise $F^2 < 0$. The (-2)-curves are grouped in non-intersecting collections each with negative definite intersection pairing.

The claim, along with the fact that $b_1(\mathcal{X}) = 2$ and the Hodge-theoretic bound on the Picard number $\rho(X_{\min}) \leq \frac{1}{6}(5c_2 - c_1^2) + b_1(\mathcal{X})$ give that $\rho_{-2}(\mathcal{X}) \leq \frac{4c_2}{5}$. It follows from Lemma 4 that

$$Lh^1_\Omega(\mathcal{X}) + \frac{s_2(\mathcal{X})}{3!} < \frac{1}{6}(\frac{4c_2}{5} + \frac{c_2}{5} - c_2) = 0,$$

and hence $\mathcal{X}$ does not satisfy the CMS-criterion.

Claim—The H-bound and M-bound cannot rule out the existence of $\mathcal{X} \in GT$ satisfying the CMS-criterion if $c_1^2 \geq \frac{1}{5}(c_2 + 7)$.

If $c_1^2 > c_2$, then $\mathcal{X}$ necessarily satisfies the CMS-criterion.

If $\frac{1}{5}(c_2 + 7) \leq c_1^2 \leq c_2$, then due to Lemma 3, the H-bound with $b_1(\mathcal{X}) = 0$ gives the lowest upper bound to $\rho_{-2}(\mathcal{X})$. Hence we can not rule out $\rho_{-2}(\mathcal{X}) = \frac{1}{6}(5c_2 - c_1^2) - 1$ (this number is an integer due to $c_2 + c_1^2 \equiv 0 \bmod 12$) and that these (-2)-curves come from a single singularity of type A in X_{can}. In this case Lemma 4 gives

$$Lh^1_\Omega(\mathcal{X}) = h^1_\Omega(A_{\frac{1}{6}(5c_2 - c_1^2) - 1}) > \frac{1}{36}(5c_2 - c_1^2 - 6) - \frac{1}{37}$$

and

$$Lh^1_\Omega(\mathcal{X}) + \frac{s_2(\mathcal{X})}{3!} < \frac{1}{36}(5c_1^2 - c_2 - 6) - \frac{1}{37}$$

and the claim follows.

The claim implies that the H-bound and M-bound cannot rule out the existence of $\mathcal{X} \in GT$ with $\Omega^1_{\mathcal{X}}$ big with any pair of Chern numbers satisfying $c_1^2 \geq \frac{1}{5}(c_2 + 7)$. Therefore. it allows

$$\alpha_{\text{Big}} = \inf\left\{\frac{c_1^2}{c_2}(\mathcal{X}) |\ \mathcal{X} \in GT \text{ with } \Omega^1_{\mathcal{X}} \text{ big}\right\}$$

to be as low as $\frac{1}{5}$.

□

4 Applications

4.1 Degrees of hypersurfaces of $\mathbb{P}^3$ with deformations with big cotangent bundle and the Green-Griffiths-Lang conjecture

Smooth hypersurfaces $H_d \subset \mathbb{P}^3$ of degree $d \geq 5$ have the cotangent bundle $\Omega^1_{H_d}$ with the positivity property that the canonical divisor $K_{H_d} = (d-4)H$ is ample (H an hyperplane section of H_d). Nevertheless, these surfaces H_d have no symmetric differentials ([22], [34] or see also [35]), so $\Omega^1_{H_d}$ is very far from being positive in the sense of being big. Hence, symmetric differentials play no role in proving the Green-Griffiths-Lang (GGL) conjecture for smooth hypersurfaces or addressing the Kobayashi Conjecture stating that a general hypersurface of $\mathbb{P}^n$ of sufficient large degree has no entire curves. In both cases, one has to use jet differentials, Diverio, Merker and Rousseau in [36] proved GGL-conjecture for generic hypersurfaces of $\mathbb{P}^n$ with sufficiently large degree and finally Brotbek proved the Kobayashi conjecture in [37] (in both cases the degrees have to be large).

We are interested in determining when symmetric differentials play a role in the GGL conjecture in the following two related cases:

1. hypersurfaces of $\mathbb{P}^3$ with canonical singularities;
2. special representatives of the deformation equivalence classes of smooth hypersurfaces of $\mathbb{P}^3$.

Recall, two smooth surfaces X and Y are said to be deformation equivalent if there are smooth pairs $(\mathcal{X}_i, T_i)$ with $\pi_i : \mathcal{X}_i \to T_i$ smooth fibrations and $t_{i,0}, t_{i,1} \in T_i$, $i = 1, ..., k$, such that $X \simeq \pi_1^{-1}(t_{1,0})$, $\pi_i^{-1}(t_{i,1}) \simeq \pi_i^{-1}(t_{i+1,0})$ and $Y \simeq \pi_1^{-1}(t_{k,1})$.

Both cases (a) and (b) are related since the minimal resolution of a hypersurface $X_d \subset \mathbb{P}^3$ of degree d with canonical singularities is deformation equivalent to a smooth hypersurface of $\mathbb{P}^3$ of the same degree d. This is a consequence of the Brieskorn simultaneous resolution theorem [38].

The tool in the background is the work of Bogomolov [3] and McQuillan [4] that state that the algebraic differential equations coming from symmetric differntials on a projective surface X are enough to obtain the GGL-conjecture if X has big Ω^1_X.

Question: Let $[H_d]$ be the deformation equivalence class of smooth hypersurfaces of $\mathbb{P}^3$ with degree d. Is

$$S := \{d \mid \exists X \in [H_d] \text{ with } \Omega^1_X \text{ big}\}$$

non-empty? If yes, what is $d_{\min} := \min S$?

The first result [9] on this question showed that $S \neq \varnothing$ using representatives of $[H_d]$ which are resolutions of nodal hypersurfaces (a miscalculation of $\hbar^0_\Omega(A_1)$ caused a wrong upper bound for $d_{\min}$). In [19] also using of nodal hypersurfaces it was shown that $d_{\min} \leq 13$ (due to Remark 2 the RR-criterion can at best obtain $d_{\min} \leq 12$). In [15] with the correct $h^1_\Omega(A_1)$, $d_{\min} \leq 10$ was achieved (see also [39]). This is the strongest result possible using nodal hypersurfaces and the CMS criterion. In [40] two of the authors showed that by considering A_2 singularities one can improve upon the nodal case and obtain $d_{\min} \leq 9$. Joint work of the second author and the third author appeared in the thesis [16] where the computation of $h^1_\Omega(A_n)$ for $n = 1, 2, 3, 4, 8, 12, 16$ and $d_{\min} \leq 8$ are achieved (Theorem 4 is the publication of this latter result).

The proof of Theorem 4 is distinct from the one in [16]. In that one it was used a known construction of an hypersurface of degree 8 with 64 A_3 singularities and the knowledge of $h^1_\Omega(A_3)$. In this proof make use of Theorem 2(II) [41], which makes possible the determination of $Lh^1_\Omega(Y)$ for all the minimal resolutions Y of hypersurfaces $X \subset \mathbb{P}^3$ with known number of singularities of type A.

Proof (of Theorem 4) Let X_d be a cyclic cover of $\mathbb{P}^2$ of degree d branched along d lines in general position. X_d is a hypersurface of $\mathbb{P}^3$ of degree d with $\frac{1}{2}d(d-1)$ A_{d-1} singularities. Denote by Y_d the minimal resolution of X_d.

Since the singular surface X_d only has canonical singularities, Brieskorn simultaneous resolution theorem [38] gives that Y_d is deformation equivalent to a smooth hypersurface of degree d in $\mathbb{P}^3$. Additionally, by Ehresmann's fibration theorem, Y_d is diffeomorphic to a smooth hypersurface of degree d. Hence, $s_2(Y_d) = d(10 - 4d)$.

The number of (-2)-curves in Y_d is $\rho_{-2}(Y_d) = \frac{d(d-1)^2}{2}$. It follows from Lemma 4 that the CMS-criterion condition

$$\frac{s_2(Y_d)}{6} + Lh^1_\Omega(Y_d) > 0$$

holds if

$$37d^2 - 376d + 783 \geq 0.$$

The above holds for all $d \geq 8$.

On the other hand, also due to Lemma 4, the CMS-criterion holds for Y_d only if $\rho_{-2}(Y_d) + s_2(Y_d) > 0$, i.e.

$$(d-7)(d-3) > 0$$

Hence it is not possible to derive the existence of deformations of hypersurfaces of degrees $d = 5, 6, 7$ with big cotangent bundle using cyclic covers of $\mathbb{P}^2$ of degree d branched along d lines in general position and the CMS-criterion. □

Following the line of the arguments in the proof above, we obtain:

Proposition 13 *Let $X \subset \mathbb{P}^3$ be a hypersurface of degree $d \geq 5$ with only canonical singularities of type A and* $\text{Sing}(X) = \{x_1, ..., x_k\}$ *with x_i of type A_{n_i}. Then X satisfies the GGL-conjecture if*

$$\sum_{i=1}^{k} n_i - \frac{6k}{37} \geq d(4d-10) \tag{3.2}$$

Proof The minimal resolution Y of X has $\rho_{-2}(Y) = \sum_{i=1}^{k} n_i$ coming from k singularities and $s_2(Y) = d(10-4d)$. It follows from Lemma 4 and the CMS-criterion that Ω_Y^1 is big if 3.2 holds.

Once bigness of the cotangent bundle Ω_Y^1 is guaranteed, [4] gives that Y satisfies the GGL-conjecture and this implies the same for X. Entire curves X lift, relative to the resolution map $\sigma : Y \to X$, to Y. This implies that the degeneracy locus of $Z_X \subset X$ (the minimal subvariety containing all entire curves of X) is contained in the σ-projection of the degeneracy locus $Z_Y \subsetneq Y$ (note that Z_X might be distinct from $\sigma(Z_Y)$, since $\text{Sing}(X) \subset \sigma(Z_Y)$ while Z_X does not necessarily contain $\text{Sing}(X)$). □

The approach to finding $d_{\min}$ using the CMS-criterion has to be accompanied by knowledge of constructions of hypersurfaces with many canonical singularities and of theoretical bounds on the number of possible canonical singularities that can occur (e.g. H-bound and M-bound). What can be said about improving on $d_{\min} \leq 8$ that follows from Theorem 4?

Remark 4 (Varchenko's spectral bound) In the case of hypersurfaces $X \subset \mathbb{P}^3$ with only canonical singularities there is one additional known constraint on the singularities coming from Varchenko's spectral bound [42]. The implications of Varchenko's spectral bound include the H-bound (48) (it follows from using the spectral interval $[1, 2]$). For good expositions on the Varchenko's spectral bound see [43] or [44, §7.1].

The Varchenko's constraint gives better upper bounds to the number of possible singularities A_n with low n in a hypersurface of given degree than the standard H-bound. For example in degree $d = 5$ it says only 31 A_1 can occur, while the H-bound allows says 44. However, as it was explained in Theorem 3, we are interested in bounding the number of (-2)-curves (having them appear from a single singularity) and in this case Varchenko's constraint just gives the H-bound.

From the perspective of Theorem 3 on the excluded range $ER(\text{CMS})$, we leave the following

Remark 5 The case $d_{\min} \leq 6$ is not ruled out by either the H-bound or the M-bound, but $d_{\min} = 5$ can not be obtained via the CMS-criterion if we are considering resolutions of hypersurfaces with only singularities of type A.

The remark follows since $c_1^2 \geq \frac{1}{5}(c_2 + 7)$ for H_d if $d = 6, 7$, while $c_1^2 \leq \frac{1}{5}c_2$ for $d = 5$. Note that

$$\frac{c_1^2}{c_2}(H_d) = \frac{d(d-4)^2}{d(d^2-4d+6)},$$

which gives $\frac{1}{11}$, $\frac{2}{9}$ and $\frac{1}{3}$ for $d = 5, 6, 7$, respectively.

We now make observations for the degrees 7, 6 and 5:

For deformations Y_d of smooth hypersurfaces $H_d \subset \mathbb{P}^3$ of degree d, the best bound for the number (-2)-curves is the the H-bound (Lemma 3), which gives:

$$\rho_{-2}(Y_d) \leq \frac{1}{3}d(2d^2 - 6d + 7) - 1 \tag{54}$$

In the discussion below Y_d are the minimal resolution of a hypersurface of degree d with only singularities of type A.

For $d = 7$, we have $s_2(Y_7) = -126$. By Lemma 4, it follows that Y_d needs at least 127 $(-2)-$curves for the CMS-criterion to imply $\Omega_{Y_7}^1$ big (they can come from 6 singularities due to Proposition 13, but no more). Note that the bound, (54), gives $\rho_{-2}(Y_7) \leq 146$. The known construction with highest $\rho_{-2}(Y_7)$ appears in Theorem 4 with $\rho_{-2}(Y_7) = 126$ (21 A_6) and is borderline on the wrong side.

For $d = 6$, we have $s_2(Y_6) = -84$, hence the same argument as above gives that Y_6 needs at least 85 (-2)-curves for the CMS-criterion to imply $\Omega^1_{Y_6}$ big, which is the maximum possible by (54). The known construction with highest $\rho_{-2}(Y_6)$ is again the one appearing in Theorem 4 with $\rho_{-2}(Y_6) = 75$ (15 A_5).

For $d = 5$, there is no hope for the CMS-criterion to apply to Y_5, as was observed in Remark 5. We note that $s_2(Y_5) = -50$, while the bound (54) gives $\rho_{-2}(Y_5) \leq 44$. We do not expect $d_{\min} = 5$ is possible, since $\frac{c_1^2}{c_2}(H_5) = \frac{1}{11} < \frac{1}{5}$. Considering hypersurfaces with singularities of type D and E shouldn't change the outcome with respect to the CMS-criterion. The path to resolve whether $d_{\min} = 5$ is attainable should use Criterion 2 (full bigness criterion, (40)), Section 3.4, and hence getting a hold on the nonlocalized component of $h^1_\Omega(\mathcal{X})$, $NLh^1_\Omega(\mathcal{X}) := \liminf_{m\to\infty} \dfrac{h^1(X_{\text{can}}, \sigma_* S^m\Omega^1_{X_{\min}})}{m^3}$, see (46).

4.1.1 Symmetric differential of a given degree in resolutions of hypersurfaces with A_n singularities

Let Y_d be the minimal resolution of a hypersurface X_d of degree d with only singularities of type A. We consider the question of what can we say about $h^0(Y_d, S^m\Omega^1_{Y_d})$, if we know the singularities of X_d.

Let $\text{Sing}(X_d) = \{x_1, ..., x_k\}$ with x_i an A_{n_i} singularity. Then, by Riemann-Roch and the lower bound for $h^1(Y_d, S^m\Omega_{Y_d})$, coming from (42), we have:

$$h^0(Y_d, S^m\Omega^1_{Y_d}) + h^2(Y_d, S^m\Omega^1_{Y_d}) \geq \chi(Y_d, S^m\Omega_{Y_d}) + \sum_{i=1}^{k} h^1(A_{n_i}, m), \tag{55}$$

where

$$\chi(Y_d, S^m\Omega_{Y_d}) = -\frac{2d^2-5d}{3}m^3 - \frac{d^3-4d^2+6d}{2}m^2 - \frac{2d^3-10d^2+17d}{6}m + \frac{d^3-6d^2+11d}{6}.$$

For $m \geq 3$, Bogomolov's vaninishing (3.1) gives $h^2(Y_d, S^m\Omega^1_{Y_d}) = 0$, hence the right side of (55) gives a lower bound for $h^0(Y_d, S^m\Omega^1_{Y_d})$.

For $m = 2$, Serre duality and $(\Omega^1_X)^\vee \simeq \Omega^1_X \otimes \mathcal{O}(-K_X)$ if X a surface give $h^2(Y_d, S^2\Omega^1_{Y_d}) = h^0(Y_d, S^2\Omega^1_{Y_d} \otimes \mathcal{O}(-K_{Y_d}))$. Moreover, since $h^0(Y_d, \mathcal{O}(K_{Y_d})) = p_g(H_d) \neq 0$, where $H_d \subset \mathbb{P}^3$ is a smooth hypersurface of degree $d \geq 5$, it follows that:

$$h^2(Y_d, S^m\Omega^1_{Y_d}) < h^0(Y_d, S^m\Omega^1_{Y_d})$$

In particular, it follows that $h^0(Y_d, S^2\Omega^1_{Y_d}) \neq 0$, if the right side of (55) is positive for $m = 2$.

We have all the ingredients to find the right side of (55). Recall that $h^1(A_{n_i}, m) = \chi(A_{n_i}, m) - \hbar^0(A_{n_i}, m)$, the first term of the right side is given by

Proposition 14 *The local Euler characteristic $\chi(A_n, m)$ is given by*

$$\chi(A_n, m) = \frac{n^2+2n}{6(n+1)}\left(m^3 + 3m^2 + \frac{3}{2}m - \frac{1}{2}\right) + \begin{cases} \frac{(-1)^m}{4}\lfloor\frac{m+1}{n+1}\rfloor + \alpha_n(m) & n \text{ odd} \\ \alpha_n(m) & n \text{ even,} \end{cases} \tag{56}$$

where $\alpha_n(m)$ is as defined in Theorem 6.

Proof By Equation (12), $\chi(A_n, m) = \mu(A_n, m) - \chi_{\text{orb}}(A_n, m)$. The then result follows from (25) and Theorem 6. □

The second term $\hbar^0(x, m)$ is determined by the formula in Theorem 1 in [41]. A relevant property of $\hbar^0(A_n, m)$ to the following discussion is that $\hbar^0(A_n, m) = \hbar^0(A_n, n)$ for $m \geq n$.

Next, we consider the minimal degrees m that are guaranteed to exist by (55) in surfaces Y'_d and Y''_d belonging to two subclasses of the surfaces Y_d. The surfaces Y'_d are realizable (the ones appearing in proof of Theorem 4) and Y''_d are not theorectically excluded. The surfaces Y''_d are the minimal resolutions of hypersurfaces X_d with a single singularity of type A_n with n the largest possible respecting the known bounds. The strongest bound in this case is the H-bound giving $n = \frac{1}{3}d(2d^2 - 6d + 7) - 1$. We consider the latter class, since it is the best possible concerning (55).

Proposition 15 —

(a) *(Realizable) Let Y'_d be the minimal resolution of $X_d \subset \mathbb{P}^3$ which is a cyclic cover of $\mathbb{P}^2$ of degree d branched along d lines in general position. Then*

$$h^0(Y'_d, S^4\Omega^1_{Y'_d}) > 0, \quad \forall d \geq 30$$

while no symmetric differentials of degrees 2, 3 on Y'_d for all $d \geq 5$ are guaranteed by (55).

(b) *(Theoretically not excluded) Let Y''_d be the minimal resolution of $X_d \subset \mathbb{P}^3$ an hypersurface with a single singularity of type $A_{\frac{1}{3}d(2d^2-6d+7)-1}$. Then*

$$h^0(Y''_d, S^2\Omega^1_{Y''_d}) > 0, \quad \forall d \geq 16$$

Proof The purpose of this result is to determine the minimal m for which the existence of symmetric differentials of degree m on the surfaces of type Y'_d and Y''_d are guaranteed by (55). It is enough to consider solely $m = 2, 3, 4$.

It follows from the formula for $\hbar^0(A_n, m)$ in Theorem 1 of [41], that $\hbar^0(A_n, m) = \hbar^0(A_n, n)$ for $m \geq n$ and $\hbar^0(A_2, 2) = 3$, $\hbar^0(A_3, 3) = 8$ and $\hbar^0(A_4, 4) = 18$. Additionally, using Proposition 14 and $h^1(A_{n_i}, m) = \chi(A_{n_i}, m) - \hbar^0(A_{n_i}, m)$, we obtain

$$h^1(A_n, m) = \begin{cases} 4n - 1, & n \geq 2 \quad \text{if } m = 2 \\ 10n - 2, & n \geq 3 \quad \text{if } m = 3 \\ 20n - 4, & n \geq 4 \quad \text{if } m = 4. \end{cases} \tag{57}$$

(a) In this case, the right side of (55) is

$$\chi(Y'_d, S^m\Omega_{Y'_d}) + \frac{d(d-1)}{2} h^1(A_{d-1}, m) = \begin{cases} -\dfrac{d(d^2-1)}{2}, & \text{if } m = 2 \\ -\frac{d^3}{3} - 7d^2 + \frac{52d}{3}, & \text{if } m = 3 \\ \dfrac{5d^3 - 162d^2 + 367d}{6}, & \text{if } m = 4. \end{cases} \tag{58}$$

For $m = 2, 3$: the right side of (55) is < 0 for all $d \geq 5$, while for $m = 4$ it is positive for $d \geq 30$.

(b) In this case, the right side of (55) for m=2 is

$$\chi(Y''_d, S^2\Omega_{Y''_d}) + h^1(A_{\frac{1}{3}d(2d^2-6d+7)-1}, 2) = \frac{1}{6}\left(d^3 - 18d^2 + 41d - 32\right). \tag{59}$$

It is positive for $d \geq 16$. □

4.2 Cyclic covers of $\mathbb{P}^2$ branched over line arrangements

We consider the minimal resolutions $Y_{n,d}$ of cyclic covers $X_{n,d}$ of degree n of $\mathbb{P}^2$ branched along d lines in general position, $d = \nu n$ (note that the X_d of the previous sub-section are now denoted by $X_{d,d}$). The surfaces $X_{n,d}$ have $\frac{d(d-1)}{2}$ A_{n-1} singularities and $Y_{n,d}$ are of general type except for the pairs $(n, d) = (2, 2), (2, 4), (2, 6), (3, 3)$ and $(4, 4)$.

Proof (of Theorem 5) To facilitate the reader we express the pairs (n, ν) for which the surfaces $Y_{n,\nu n}$ are not of the general type: $(2,1)$, $(2,2)$, $(2,3)$, $(3,1)$ and $(4,1)$. Hence they are excluded from the table above.

The 2nd Segre number of $Y_{n,d}$ and the localized component $Lh^1_\Omega(Y_{n,d})$, by Lemma 4, satisfy

$$s_2(Y_{n,d}) = \left(\frac{1}{n} - 1\right) d^2 - 3(n-1)d + 6n$$

$$\frac{1}{6}\frac{d(d-1)}{2}\left(n - 1 - \frac{1}{37}\right) < Lh^1_\Omega(Y_{n,d}) < \frac{1}{6}\frac{d(d-1)}{2}(n-1)$$

The CMS criterion gives that $Y_{\ell,d}$ has big cotangent bundle if

$$\frac{d(d-1)}{2}\left(n - 1 - \frac{6}{37}\right) \geq \left(1 - \frac{1}{n}\right)\ell^2 + 3(n-1)d - 6n$$

and only if

$$\frac{d(d-1)}{2}(n-1) > \left(1 - \frac{1}{n}\right) d^2 + 3(n-1)d - 6n.$$

The table follows from the above. Note that the CMS-criterion will never hold on double covers ($n = 2$). $\square$

Remark 6 To compare with Theorem 15 [19] that uses the RR-criterion, we give the table in a form to make clear the difference of the pairs excluded with both criteria.

Table 3: A comparison of the CMS and RR criteria for the pairs (n, ν) for which $\Omega^1_{Y_{n,\nu n}}$ is not big

	n	2	3	4	5	6	7	8	$9 \leq n \leq 14$
CMS	ν	≥ 1	≤ 7	≤ 3	1	1	1		
RR	ν	≥ 1	≥ 1	≤ 12	≤ 6	≤ 3	≤ 2	≤ 2	1

The striking difference lies in the case of covers of degree $n = 3$ for which the CMS-criterion includes infinitely more pairs with big cotangent bundle. The case of $\nu = 1$ and concerning resolutions of hypersurfaces in $\mathbb{P}^3$ of degree n is also striking since one has the improvement from $n \leq 14$ to $n \leq 7$.

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